

Problem 1 (4 p.): A physical system is in state $|\alpha\rangle$ when a measurement of the quantised observable A is performed. Discuss the basic aspects of this, considering resulting values and states. Also, obtain an expression for the expectation value and discuss it.

Solution: Observable A is an operator. Applying A corresponds to the measurement of a physical quantity and the only possible results of the measurement are the quantized eigenvalues $a^{(i)}$ of A . Since the eigenstates $|a^{(i)}\rangle$ of A form a basis, the state $|\alpha\rangle$ can be expanded (using the completeness relation) as $|\alpha\rangle = \sum_i |a^{(i)}\rangle \langle a^{(i)}|\alpha\rangle = \sum_i c_i |a^{(i)}\rangle$. The probability for result $a^{(i)}$ at a measurement is $P = |c_i|^2 = |\langle a^{(i)}|\alpha\rangle|^2$. The measurement affects the state $|\alpha\rangle$, which goes ('collapses') into the eigenstate $|a^{(i)}\rangle$ corresponding to the measured eigenvalue $a^{(i)}$, i.e. one term in the expansion $|\alpha\rangle = \sum_i c_i |a^{(i)}\rangle$ is selected.

Determination of the probability $|c_i|^2$ requires many measurements on identical systems (ensemble), i.e. prepared in state $|\alpha\rangle$. The expectation value of A with respect to $|\alpha\rangle$ is then the mean value of many such measurements. Formally, $\langle A \rangle = \langle \alpha|A|\alpha\rangle = \sum_{i,j} \langle \alpha|a^{(i)}\rangle \langle a^{(i)}|A|a^{(j)}\rangle \langle a^{(j)}|\alpha\rangle = \sum_i a^{(i)} |\langle a^{(i)}|\alpha\rangle|^2$, i.e. a sum over the eigenvalues weighted by their associated probabilities in the measurement. This implies that the expectation value is not quantised, as opposed to the eigenvalues.

Problem 2 (4 p.): The observable A has eigenstates $|1\rangle$ and $|2\rangle$ and the hamiltonian operator is $H = C(|1\rangle\langle 2| + |2\rangle\langle 1|)$, where C is a constant.

- (a) Derive the energy eigenstates and their eigenvalues.
 (b) For a system in state $|1\rangle$ at $t = 0$, find the state vector (in Schrödinger picture) for $t > 0$ and the corresponding probability for it to be in state $|2\rangle$.
 (c) What physical situation can this describe? What is then A , H and C ?

Solution:

(a) The eigenstates of the observable A , $|i\rangle$ with $i = 1, 2$, are orthonormal $\langle i|i'\rangle = \delta_{ii'}$ and form a complete basis. These states are not eigenstates to the hamiltonian H , since

$$\begin{aligned} H|1\rangle &= C(|1\rangle\langle 2| + |2\rangle\langle 1|)|1\rangle = C|1\rangle\langle 2|1\rangle + C|2\rangle\langle 1|1\rangle = C|2\rangle, \\ H|2\rangle &= C(|1\rangle\langle 2| + |2\rangle\langle 1|)|2\rangle = C|1\rangle\langle 2|2\rangle + C|2\rangle\langle 1|2\rangle = C|1\rangle. \end{aligned}$$

The eigenstates $|E\rangle$ to H can be written $|E\rangle = a|1\rangle + b|2\rangle$, with $|a|^2 + |b|^2 = 1$. The eigenvalue equation $H|E\rangle = E|E\rangle$ then gives, for the right-hand and left-hand sides

$$\begin{aligned} H|E\rangle &= C(|1\rangle\langle 2| + |2\rangle\langle 1|)(a|1\rangle + b|2\rangle) = C(a|2\rangle + b|1\rangle) \\ E|E\rangle &= E(a|1\rangle + b|2\rangle). \end{aligned}$$

Equating the coefficients for $|1\rangle$ and $|2\rangle$ gives $E = \pm C$ and $a = \pm b$, i.e. after normalisation the solution

$$\begin{aligned} |E+\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \quad \text{with eigenvalue } C, \\ |E-\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle), \quad \text{with eigenvalue } -C. \end{aligned}$$

Alternatively, this could have been obtained using the matrix representations $|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|2\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $H \doteq \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$ and solving $\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix}$.

(b) The initial state, at time $t = 0$, is $|\alpha(0)\rangle = |1\rangle$, while at a later time $t > 0$, the state of the system is given by the action of the time-evolution operator $|\alpha(t)\rangle = \mathcal{U}(t)|\alpha(0)\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha(0)\rangle$. Expressing the vectors $|i\rangle$ in terms of the basis $\{|E+\rangle, |E-\rangle\}$ the state of the system becomes

$$\begin{aligned} |\alpha(t)\rangle &= e^{-\frac{i}{\hbar}Ht}|1\rangle = e^{-\frac{i}{\hbar}Ht} \frac{1}{\sqrt{2}}(|E+\rangle + |E-\rangle) = \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Ht}|E+\rangle + \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Ht}|E-\rangle = \\ &= \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}Ct}|E+\rangle + \frac{1}{\sqrt{2}}e^{\frac{i}{\hbar}Ct}|E-\rangle = \frac{1}{2}e^{-\frac{i}{\hbar}Ct}(|1\rangle + |2\rangle) + \frac{1}{2}e^{\frac{i}{\hbar}Ct}(|1\rangle - |2\rangle) = \\ &= \cos\left(\frac{Ct}{\hbar}\right)|1\rangle - i \sin\left(\frac{Ct}{\hbar}\right)|2\rangle. \end{aligned}$$

The probability of finding the system in state $|2\rangle$ is then $\mathcal{P}(t) = |\langle 2|\alpha(t)\rangle|^2 = \sin^2\left(\frac{Ct}{\hbar}\right)$.

(c) Since we have a two-state system, one first considers a spin-1/2 particle. The operator A then corresponds to S_z with eigenstates $\{|1\rangle, |2\rangle\} = \{|+\rangle, |-\rangle\}$. H is seen to be essentially S_x , or the Pauli matrix σ_x and can then describe the interaction with an external, static magnetic field \mathbf{B} along the x direction, i.e. $H = -\left(\frac{e}{mc}\right)\mathbf{S}\cdot\mathbf{B} = -\left(\frac{eB}{mc}\right)S_x = \omega S_x$, where ω is the precession frequency, and $S_x = \frac{\hbar}{2}\sigma_x$ with σ_x the first Pauli matrix. Thus, the eigenvalues of the hamiltonian H are $\pm C = \pm\frac{\hbar\omega}{2}$.

Problem 3 (4 p.): Add the angular momenta $j_1 = 1/2$ and $j_2 = 1/2$ to form all possible states $|j, m\rangle$, of total angular momentum j , expressed as linear combinations of $|j_1 j_2; m_1 m_2\rangle$. This should be done through an explicit derivation using ladder operators and not only reading from the table of Clebsch-Gordan coefficients.

Solution: Using completeness relation, express the new states as a linear combination of the old direct product basis states $|j, m\rangle = \sum_{m_1 m_2} |j_1 j_2; m_1 m_2\rangle \langle j_1 j_2; m_1 m_2 | j, m\rangle$, where the scalar product is the Clebsch-Gordan (CG) coefficients. The new j can take the values $|j_1 - j_2| \leq j \leq j_1 + j_2$, i.e. for $j_1 = \frac{1}{2}, j_2 = \frac{1}{2}$ one gets $j = 0$, with $m = 0$, and $j = 1$ with $m = 0, 1$. The "highest" state can only be obtained in one way, $|j = 1, m = 1\rangle = +1|j_1 = \frac{1}{2}, j_2 = \frac{1}{2}; m_1 = +\frac{1}{2}, m_2 = +\frac{1}{2}\rangle = |+\frac{1}{2}, +\frac{1}{2}\rangle$ with the CG-coefficient +1 as phase convention and short-hand notation $|m_1, m_2\rangle$.

Now apply the ladder operator $J_- = J_{1-} + J_{2-}$ on both sides $J_-|j = 1, m = 1\rangle = (J_{1-} + J_{2-})|m_1 = +\frac{1}{2}, m_2 = +\frac{1}{2}\rangle$ and using $J_{\pm}|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$ gives $\hbar\sqrt{(1+1)(1-1+1)}|j = 1, m = 0\rangle = \hbar\sqrt{\left(\frac{1}{2} + \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2} + 1\right)}\left(|-\frac{1}{2}, +\frac{1}{2}\rangle + |+\frac{1}{2}, -\frac{1}{2}\rangle\right)$, i.e. $|j = 1, m = 0\rangle = \frac{1}{\sqrt{2}}\left(|+\frac{1}{2}, -\frac{1}{2}\rangle + |-\frac{1}{2}, +\frac{1}{2}\rangle\right)$. On this, apply ladder operators in the same way $J_-|j = 1, m = 0\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}\left(|+\frac{1}{2}, -\frac{1}{2}\rangle + |-\frac{1}{2}, +\frac{1}{2}\rangle\right)$ giving $\hbar\sqrt{2}|j = 1, m = -1\rangle = \hbar\frac{2}{\sqrt{2}}|-\frac{1}{2}, -\frac{1}{2}\rangle$, i.e. $|j = 1, m = -1\rangle = |-\frac{1}{2}, -\frac{1}{2}\rangle$. These triplet states are symmetric under the exchange $m_1 \leftrightarrow m_2$

The remaining state $|j = 0, m = 0\rangle$ must have $m = m_1 + m_2 = 0$, i.e. $m_1 = -m_2$, and be orthogonal to the state $|j = 1, m = 0\rangle$ above, and be normalized, resulting in $|j = 0, m = 0\rangle = \frac{1}{\sqrt{2}} \left(|+\frac{1}{2}, -\frac{1}{2}\rangle - |-\frac{1}{2}, +\frac{1}{2}\rangle \right)$. This singlet state is anti-symmetric under the exchange $m_1 \leftrightarrow m_2$.

Problem 4 (4 p.): A two-dimensional harmonic oscillator has the Hamiltonian $H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2} (x^2 + y^2)$.

- (a) What are the energies and eigenstates for the ground level and the first excited level?
 (b) A perturbation $V = \epsilon m\omega^2 xy$ is applied. For the states in (a), find the corresponding zeroth-order energy eigenstates and their energies to first order.

Solution:

(a) Since the oscillations in x and y are independent, the direct product states are $|n_x, n_y\rangle$ with energy eigenvalues $E_{n_x, n_y} = E_{n_x} + E_{n_y} = \hbar\omega(n_x + n_y + 2\frac{1}{2})$. The ground state is $|n_x = 0, n_y = 0\rangle = |0, 0\rangle$ with energy $E_{00} = \hbar\omega$, i.e. a nondegenerate state. The first excited states are $|1, 0\rangle$ with $E_{10} = 2\hbar\omega$ and $|0, 1\rangle$ with $E_{01} = 2\hbar\omega$, i.e. 2-fold degeneration.

(b) The perturbation $V = \epsilon m\omega^2 xy$ gives $H = H_0 + V$.

For the ground state, apply nondegenerate perturbation theory: zeroth order eigenstate is $|0, 0\rangle$ with energy $E_{00} = E_{00}^{(0)} + \Delta E_{00}^{(1)} = \hbar\omega$, since the first order energy shift $\Delta E_{00}^{(1)} = \epsilon m\omega^2 \langle 0, 0 | xy | 0, 0 \rangle = \langle 0 | x | 0 \rangle \langle 0 | y | 0 \rangle = 0$ obtained with $\langle i | j \rangle = \delta_{ij}$ and (analogously for y) $x = \sqrt{\frac{\hbar}{2m\omega}}(a_x + a_x^\dagger) \Rightarrow \langle 0 | x | 0 \rangle = 0$; $\langle 1 | x | 1 \rangle = 0$; $\langle 1 | x | 0 \rangle = \langle 0 | x | 1 \rangle = \sqrt{\frac{\hbar}{2m\omega}}$.

For the first excited states, apply degenerate perturbation theory: zeroth order states and first order energy shifts are obtained by diagonalising the matrix representation of the perturbing operator, i.e.

$$V \doteq \begin{pmatrix} \langle 10 | V | 10 \rangle & \langle 10 | V | 01 \rangle \\ \langle 01 | V | 10 \rangle & \langle 01 | V | 01 \rangle \end{pmatrix} = \epsilon m\omega^2 \begin{pmatrix} 0 & \langle 1 | x | 0 \rangle \langle 0 | y | 1 \rangle \\ \langle 0 | x | 1 \rangle \langle 1 | y | 0 \rangle & 0 \end{pmatrix} = \epsilon \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

using position operators x and y as in (a). Thus, solve the eigenvalue equation

$$\epsilon \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle 10 | \ell^{(0)} \rangle \\ \langle 01 | \ell^{(0)} \rangle \end{pmatrix} = \Delta E^{(0)} \begin{pmatrix} \langle 10 | \ell^{(0)} \rangle \\ \langle 01 | \ell^{(0)} \rangle \end{pmatrix} \Leftrightarrow \begin{pmatrix} -\Delta E^{(0)} & \epsilon \frac{\hbar\omega}{2} \\ \epsilon \frac{\hbar\omega}{2} & -\Delta E^{(0)} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

where $a = \langle 10 | \ell^{(0)} \rangle$ and $b = \langle 01 | \ell^{(0)} \rangle$. Equating the determinant of the last 2×2 matrix to zero, gives $\Delta E^{(0)} = \pm \epsilon \frac{\hbar\omega}{2}$. Inserting these solutions back into the matrix equation

gives $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$ resulting in $a = 1/\sqrt{2}$, $b = \pm 1/\sqrt{2}$ using normalisation

$|a|^2 + |b|^2 = 1$. Thus, the perturbed states to zeroth order with energies to first order are $|\ell_{\pm}^{(0)}\rangle = \frac{1}{\sqrt{2}} (|10\rangle \pm |01\rangle)$ with $E_1 = (2 \pm \frac{\epsilon}{2})\hbar\omega$.

Problem 5 (4 p.):

(a) Show that the differential cross section $d\sigma/d\Omega$ for elastic scattering of particles with mass m and energy $E = \hbar^2 k^2/2m$ on a spherical δ -shell potential with radius R and strength V_0 , in the Born approximation is given by $d\sigma/d\Omega = (2mV_0/\hbar^2)^2 [R \sin(qR)/q]^2$, where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transfer.

(b) Calculate the S -wave contribution to the scattering amplitude for the case in (a).

Hints: For a spherically symmetric potential, the scattering amplitude $f_k(\theta, \phi) = f_k(\theta)$.

$$\int P_l(\cos \theta) P_l(\cos \theta) d\Omega = 4\pi \delta_{ll'}/(2l+1); P_0(\cos \theta) = 1;$$

$$\int \sin(a\sqrt{1-x})/(a\sqrt{1-x}) dx = 2 \cos(a\sqrt{1-x})/a^2.$$

Solution:

(a) The potential is written $V(\mathbf{r}) = V_0 \delta(r - R)$, and $\frac{d\sigma}{d\Omega} = |f_k(\theta, \phi)|^2 \stackrel{\text{Born approx.}}{\approx} |f^{(1)}(\theta)|^2$.

$$\begin{aligned} f^{(1)}(\theta) &= -\frac{2m}{4\pi\hbar^2} \int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V(\mathbf{r}) d\mathbf{r} = -\frac{2mV_0}{4\pi\hbar^2} \int e^{iqr \cos \theta} \delta(r - R) r^2 dr d\phi d(\cos \theta) \\ &= -\frac{2mV_0}{4\pi\hbar^2} 2\pi R^2 \left[\frac{e^{iqRx}}{iqR} \right]_{-1}^1 = -\frac{2mV_0}{4\pi\hbar^2} 4\pi R^2 \frac{\sin qR}{qR} \end{aligned}$$

$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\hbar^2} \right)^2 R^2 \frac{\sin^2 qR}{q^2}$ in the Born approximation.

(b) Using that $f(\theta) = \sum_l (2l+1) f_l P_l(\cos \theta)$ and $\int P_l(\cos \theta) P_{l'}(\cos \theta) d\Omega = 4\pi \delta_{ll'}/(2l+1)$, one finds that $f_0 = \int f(\theta) P_0(\cos \theta) d\Omega / (4\pi)$, where $P_0(x) = 1$. We also notice that $q = 2k\sqrt{1 - \cos \theta}$. In the Born approximation we, thus, have the $l = 0$ (S -wave) contribution

$$\begin{aligned} f_0 &= \int f^{(1)}(\theta) P_0(\cos \theta) \frac{d\Omega}{4\pi} = -\frac{2mV_0}{4\pi\hbar^2} R^2 \int \frac{\sin kR \sqrt{2(1 - \cos \theta)}}{kR \sqrt{2(1 - \cos \theta)}} d\phi d(\cos \theta) \\ &= -\frac{2mV_0}{2\hbar^2} R^2 \left[\frac{2 \cos kR \sqrt{2(1 - \cos \theta)}}{(\sqrt{2}kR)^2} \right]_{-1}^1 = -\frac{mV_0}{\hbar^2} \frac{1 - \cos 2kR}{k^2} = -\frac{2mV_0}{\hbar^2} \frac{\sin^2 kR}{k^2} \end{aligned}$$

Problem 6 (4 p.):

(a) Suppose that we encode a spin $1/2$ system with information in the three different states $|s_z, +\rangle$, $|s_z, -\rangle$, and $|s_x, -\rangle$. Discuss in terms of the Basic Decoding Theorem why this can be a problem.

(b) Suppose that Alice and Bob share the Bell state $|\Psi_-\rangle = (|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2}$, and Alice wants to teleport the message $|\psi\rangle = \alpha|0\rangle_0 + \beta|1\rangle_0$. Alice performs a Bell measurement on the part of the three qubit state $|\Gamma\rangle = |\psi\rangle_0 \otimes |\Psi_-\rangle_{AB}$ of her possession, and obtains the result $|\Phi_+\rangle = (|00\rangle_{0A} + |11\rangle_{0A})/\sqrt{2}$. Determine the local unitary transformation Bob has to perform on the part of $|\Gamma\rangle$ of his possession in order to read out the message $|\psi\rangle_B$.

Hint: $\sigma^x = |1\rangle\langle 0| + |0\rangle\langle 1|$, $\sigma^y = -i|1\rangle\langle 0| + i|0\rangle\langle 1|$, $\sigma^z = |1\rangle\langle 1| - |0\rangle\langle 0|$; $P_E \geq d/N$.

Solution:

(a) The Hilbert space devoted for the information is two-dimensional, $d = 2$, while there are three messages, $N = 3$. Therefore, from the BDT, we find that the probability for an erroneous decoding of the message is $P_E \geq 1 - 2/3 = 1/3$, which means that there *will* be errors in the decoding.

(b) The conditional state that remains after Alice's measurement is given by

$$\begin{aligned} \langle \Phi_+ | \Gamma \rangle &= \frac{1}{\sqrt{2}} \left({}_{0A}\langle 00| + {}_{0A}\langle 11| \right) \frac{1}{\sqrt{2}} \left(\alpha |001\rangle_{0AB} - \alpha |010\rangle_{0AB} + \beta |101\rangle_{0AB} - \beta |110\rangle_{0AB} \right) \\ &= \frac{1}{2} \left(\alpha |1\rangle_B - \beta |0\rangle_B \right) = \frac{i}{2} \left(\alpha \left[-i|1\rangle\langle 0| + i|0\rangle\langle 1| \right] |0\rangle + \beta \left[-i|1\rangle\langle 0| + i|0\rangle\langle 1| \right] |1\rangle \right) \\ &= \frac{i}{2} \sigma^y |\psi\rangle, \text{ which means that the transformation we are looking for is } \sigma^y. \end{aligned}$$