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Exam 2011-12-16

1FA352: Quantum Mechanics 10 hp

2011 fall semester: Physics Master & Engineering Physics

Time:	8–13, i.e. 5 hours, Polacksbacken, skrivsal
Allowed aids:	Physics Handbook, Mathematics Handbook, Tashenbuch der Mathematik, enclosed collection of formulae, calculator.
Instructions:	- write legible (skriv läsligt), define symbols, motivate equations etc. - write your personal exam code on each sheet - at most one problem per sheet - put your solutions in number order in this cover

Good luck !

Note: Results available on the course web page on December 22nd.

Personal exam code:

Number of sheets per problem:

1	2	3	4	5	6

Results:

1	2	3	4	5	6

Sum:

Grade:

1. A physical system is in state $|\alpha\rangle$ when a measurement of the quantised observable A is performed. Discuss the basic aspects of this, considering resulting values and states. Also, obtain an expression for the expectation value and discuss it. (4 p.)

2. The observable A has eigenstates $|1\rangle$ and $|2\rangle$ and the hamiltonian operator is $H = C\,(|1\rangle\langle 2| \,+\, |2\rangle\langle 1|)$, where C is a constant.
   (a) Derive the energy eigenstates and their eigenvalues.
   (b) For a system in state $|1\rangle$ at $t = 0$, find the state vector (in Schrödinger picture) for $t > 0$ and the corresponding probability for it to be in state $|2\rangle$.
   (c) What physical situation can this describe? What is then A, H and C ? (4 p.)

3. Add the angular momenta $j_1 = 1/2$ and $j_2 = 1/2$ to form all possible states $|j, m\rangle$, of total angular momentum j, expressed as linear combinations of $|j_1 j_2; m_1 m_2\rangle$. This should be done through an explicit derivation using ladder operators and not only reading from the table of Clebsch-Gordan coefficients. (4 p.)

4. A two-dimensional harmonic oscillator has the Hamiltonian $H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}\,(x^2 + y^2)$.
   (a) What are the energies and eigenstates for the ground level and the first excited level?
   (b) A perturbation $V = \epsilon m\omega^2 xy$ is applied. For the states in (a), find the corresponding zeroth-order energy eigenstates and their energies to first order. (4 p.)

5. (a) Show that the differential cross section $d\sigma/d\Omega$ for elastic scattering of particles with mass m and energy $E = \hbar^2 k^2/2m$ on a spherical δ-shell potential with radius R and strength V_0, in the Born approximation is given by $d\sigma/d\Omega = (2mV_0/\hbar^2)^2[R\sin(qR)/q]^2$, where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transfer.
   (b) Calculate the S-wave contribution to the scattering amplitude for the case in (a).
   Hints: For a spherically symmetric potential, the scattering amplitude $f_k(\theta, \phi) = f_k(\theta)$.
   $\int P_l(\cos\theta) P_{l'}(\cos\theta) d\Omega = 4\pi\delta_{ll'}/(2l+1)$; $P_0(\cos\theta) = 1$;
   $\int \sin(a\sqrt{1-x})/(a\sqrt{1-x})dx = 2\cos(a\sqrt{1-x})/a^2$. (4 p.)

6. (a) Suppose that we encode a spin 1/2 system with information in the three different states $|s_z, +\rangle$, $|s_z, -\rangle$, and $|s_x, -\rangle$. Discuss in terms of the Basic Decoding Theorem why this can be a problem.
   (b) Suppose that Alice and Bob share the Bell state $|\Psi_-\rangle = (|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2}$, and Alice wants to teleport the message $|\psi\rangle = \alpha|0\rangle_0 + \beta|1\rangle_0$. Alice performs a Bell measurement on the part of the three qubit state $|\Gamma\rangle = |\psi\rangle_0 \otimes |\Psi_-\rangle_{AB}$ of her possession, and obtains the result $|\Phi_+\rangle = (|00\rangle_{0A} + |11\rangle_{0A})/\sqrt{2}$. Determine the local unitary transformation Bob has to perform on the part of $|\Gamma\rangle$ of his possession in order to read out the message $|\psi\rangle_B$.
   Hint: $\sigma^x = |1\rangle\langle 0| + |0\rangle\langle 1|$, $\sigma^y = -i|1\rangle\langle 0| + i|0\rangle\langle 1|$, $\sigma^z = |1\rangle\langle 1| - |0\rangle\langle 0|$; $P_E \geq 1 - d/N$. (4 p.)

Collection of formulae

Angular momentum: $J_\pm = J_x \pm iJ_y$ $\quad J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$

Addition: $|j_1,j_2;j=j_1+j_2,m=j\rangle = |j_1,j_2;m_1=j_1,m_2=j_2\rangle$

$[x_i,F(\bar{p})] = i\hbar\frac{\partial F}{\partial p_i}$; $[p_i,G(\bar{x})] = -i\hbar\frac{\partial G}{\partial x_i}$

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ $\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Harmonic oscillator: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$ $\quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$ $\quad [a,a^\dagger]=1$ $\quad N = a^\dagger a$

$a|n\rangle = \sqrt{n}|n-1\rangle$ $\quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

Time-independent perturbation theory:
Non-degenerate eigenvalue
$|n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)}-E_k^{(0)}} + \cdots$; $\Delta_n \equiv E_n - E_n^{(0)} = \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)}-E_k^{(0)}} + \cdots$

Degenerate eigenvalue
$|\ell\rangle = |\ell^{(0)}\rangle + \lambda|\ell^{(1)}\rangle + \cdots$ $\quad \Delta_\ell = \lambda\Delta_\ell^{(1)} + \lambda^2\Delta_\ell^{(2)} + \cdots$

$$\begin{pmatrix} V_{11} & V_{12} & \cdots \\ V_{21} & V_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix} = \Delta_\ell^{(1)}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix}$$

Time dependent perturbation theory:

$$\begin{aligned} c_n^{(0)}(t) &= \delta_{ni} \\ c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = \frac{-i}{\hbar}\int_{t_0}^{t}e^{i\omega_{ni}t'}V_{ni}(t')dt' \\ \omega_{ni} &= \frac{E_n - E_i}{\hbar} \end{aligned}$$

The Golden Rule: $w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i)$

The scattering amplitude: $f(\mathbf{k}',\mathbf{k}) = -\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\langle\mathbf{k}'|T|\mathbf{k}\rangle \overset{V=V(r)}{=} \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta)$

Perturbative expansion: $T = V + VG^+V + \ldots$

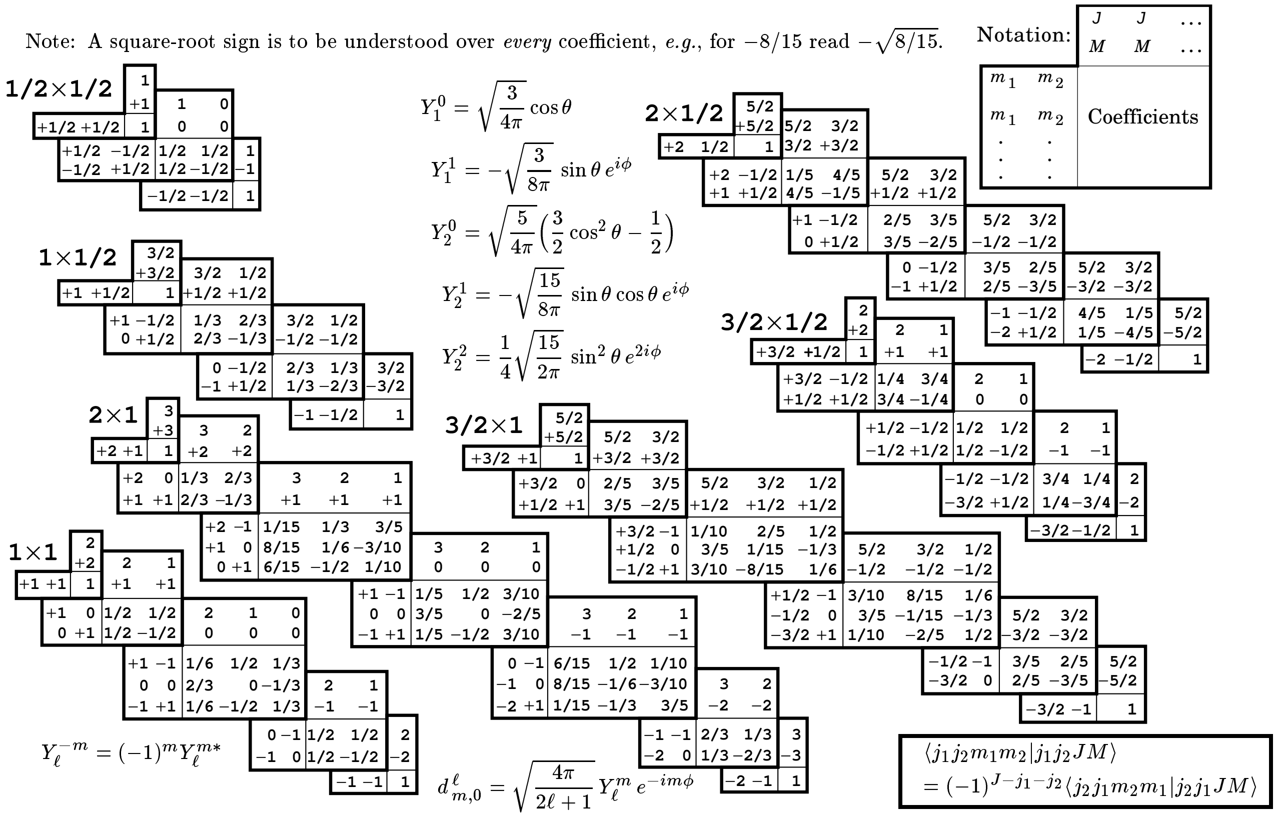
Partial wave amplitude: $f_l(k) = \frac{1}{2ik}(e^{2i\delta_l}-1) = \frac{1}{k}e^{i\delta_l}\sin\delta_l = \frac{1}{k\cot\delta_l - ik}$

Total elastic scattering cross section: $\sigma_{tot} = \int|f(\mathbf{k}',\mathbf{k})|^2 d\Omega \overset{V=V(r)}{=} 4\pi\sum_{l=0}^{\infty}(2l+1)|f_l(k)|^2$

Plane wave: $\langle\mathbf{x}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}$, $e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\hat{k}\cdot\hat{r})$

Clebsch-Gordan coefficients and spherical harmonics

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.



$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

Spherical Bessel and Neumann functions

$$j_l(kr) \overset{r\to\infty}{\to} \frac{1}{2ik}\left[\frac{e^{i(kr-l\pi/2)}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r}\right]$$

$$j_l(kr) \overset{kr\to 0}{\to} \frac{2^l l!}{(2l+1)!}(kr)^l \qquad j_0(kr) = \frac{\sin(kr)}{kr}$$

$$n_l(kr) \overset{kr\to 0}{\to} -\frac{(2l)!}{2^l l!}\frac{1}{(kr)^{l+1}} \qquad n_0(kr) = -\frac{\cos(kr)}{kr}$$

Scattering length: $a \equiv a_0 = -\lim_{k\to 0}\frac{\tan\delta_0}{k}$

For inelastic scattering we have $d\sigma(0\to n) = w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]}/j_{in}$ where the transition rate is given by

$$w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]} = \frac{2\pi}{\hbar}\,|\langle\mathbf{k}',n|V|\mathbf{k},0\rangle|^2\left(\frac{L}{2\pi}\right)^3\left(\frac{mk'}{\hbar^2}\right)d\Omega$$

at box normalisation, $\langle\mathbf{x}|\mathbf{k},n\rangle = \frac{1}{L^{3/2}}\,e^{i\mathbf{k}\cdot\mathbf{x}}|n\rangle$.

$$\int d^3x\,\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{r} = \frac{4\pi}{q^2}$$