Problem 1 (4 p.):
A beam of unpolarised spin 1/2 particles goes through a series of Stern-Gerlach magnets: the first with field along $\hat{z}$ and a slit selecting spin up, the second with field along $\hat{n}$ having the angle θ to $\hat{z}$ in the xz plane and a slit selecting spin up, the third with field along $\hat{z}$ and a slit selecting spin down. What is the intensity of the final beam relative to the incoming beam?
<u>Hints</u>: The operator $|S_n;+\rangle\langle S_n;+|$ projects out the state with spin up along $\hat{n}$.
$S_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)$, $S_y = i\frac{\hbar}{2}\left(|-\rangle\langle +| - |+\rangle\langle -|\right)$, $S_z = \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right)$.

Solution: Initial unpolarized, normalized state $|\alpha\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$. $\hat{n} = \hat{x}\sin\theta + \hat{z}\cos\theta$ and $\vec{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$ gives $S_n = \vec{S}\cdot\hat{n} = S_x\sin\theta + S_z\cos\theta = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)\sin\theta + \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right)\cos\theta$. Stern-Gerlach measurements: $M_1 = |+\rangle\langle +|$, $M_2 = |S_n+\rangle\langle S_n+|$, $M_3 = |-\rangle\langle -|$. Find $|S_n+\rangle$, i.e. solve eigenvalue equation $S_n|S_n+\rangle = \frac{\hbar}{2}|S_n+\rangle$ using Ansatz $|S_n+\rangle = a|+\rangle + b|-\rangle$ with normalisation condition $|a|^2 + |b|^2 = 1$:
$S_n|S_n+\rangle = \frac{\hbar}{2}\left[\left(|+\rangle\langle -| + |-\rangle\langle +|\right)\sin\theta + \left(|+\rangle\langle +| - |-\rangle\langle -|\right)\cos\theta\right]\left(a|+\rangle + b|-\rangle\right) = \ldots$
$= \frac{\hbar}{2}\left[\left(b\sin\theta + a\cos\theta\right)|+\rangle + \left(a\sin\theta - b\cos\theta\right)|-\rangle\right] = \frac{\hbar}{2}\left(a|+\rangle + b|-\rangle\right)$, such that equating the coefficients in front of $|+\rangle$ and $|-\rangle$ gives $a = \cos\frac{\theta}{2}$ and $b = \sin\frac{\theta}{2}$.

Applying the total measurement operator on the initial state then gives the final state $|\beta\rangle = M_3M_2M_1|\alpha\rangle = |-\rangle\langle -|\left(a|+\rangle + b|-\rangle\right)\left(a\langle +| + b\langle -|\right)|+\rangle\langle +|\frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle) = \ldots = \frac{1}{\sqrt{2}}ab|-\rangle = \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|-\rangle$. Thus, the intensity is $I = \langle\beta|\beta\rangle = \frac{1}{2}\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = \frac{1}{8}\sin^2\theta$.
Check: Intensity $I \geq 0$, maximum $I = 1/8$ for $\theta = \pi/2$ and minimum $I = 0$ for $\theta = 0$ which makes sense physically.

Problem 2 (4 p.):
For an arbitrary energy eigenstate $|n\rangle$ of a harmonic oscillator, calculate the expectation values of x^2 and p^2. What physical information can you obtain/calculate from this?

Solution:
For calculating $\langle n|x^2|n\rangle$ and $\langle n|p^2|n\rangle$ express x^2 and p^2 in terms of annihilation and creation operators: $x^2 = \left(\sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\right)^2 = \frac{\hbar}{2m\omega}\left(a^2 + aa^\dagger + a^\dagger a + {a^\dagger}^2\right)$, $p^2 = \left(-i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger)\right)^2 = -\frac{m\omega\hbar}{2}\left(a^2 - aa^\dagger - a^\dagger a + {a^\dagger}^2\right)$. Thus, we need the matrix elements: $\langle n|a^2|n\rangle = \sqrt{\cdots}\sqrt{\cdots}\langle n|n-2\rangle = 0$, $\langle n|{a^\dagger}^2|n\rangle = \sqrt{\cdots}\sqrt{\cdots}\langle n|n+2\rangle = 0$, $\langle n|a^\dagger a|n\rangle = \sqrt{n}\langle n|a^\dagger|n-1\rangle = \sqrt{n}\sqrt{n}\langle n|n\rangle = n$, $\langle n|aa^\dagger|n\rangle = \sqrt{n+1}\langle n|a|n+1\rangle = \sqrt{n+1}\sqrt{n+1}\langle n|n\rangle = n+1$. This gives $\langle n|x^2|n\rangle = \frac{\hbar}{m\omega}(n+\frac{1}{2})$ and $\langle n|p^2|n\rangle = m\omega\hbar(n+\frac{1}{2})$.

From this one obtains the expectation value of the energy: $\langle H\rangle = \frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle = \cdots = \hbar\omega\left(n + \frac{1}{2}\right)$ which is the well known energy of a harmonic oscillator. Another possibility is to calculate the uncertainties $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \langle x^2\rangle$ and $\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \langle p^2\rangle$ since $\langle x\rangle = 0$ and $\langle p\rangle = 0$ from $x, p \sim (a \pm a^\dagger)$. This gives $\sqrt{\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle} = \hbar\left(n + \frac{1}{2}\right)$, with minimum $\hbar/2$ for the ground state, and to be compared with Heisenberg's uncertainty relation $\Delta x\Delta p \geq \hbar/2$.

Problem 3 (4 p.):
An electron is in a d-orbital ($\ell = 2$). Give, with motivations/explanations, all possible states $|jm\rangle$ of total angular momentum j of the electron, expressed in terms of its spin and orbital angular momentum.

Solution:
Add orbital angular momentum $\ell = 2$ ($m_\ell = -2, -1, 0, 1, 2$) and spin angular momentum $s = 1/2$ ($m_s = -1/2, 1/2$) to total angular momentum $j = \ell - s, \ldots, \ell + s = 3/2, 5/2$ ($m_j = -j, -j+1, \ldots j-1, j$ for $j = 3/2$ and $j = 5/2$, resp.) and $m_j = m_\ell + m_s$. Express new states (eigenstates to L^2, S^2, J^2, J_z) as linear combinations in old direct product basis (eigenstates to L^2, S^2, L_z, S_z) using change of basis via completeness relation: $|\ell, s; j, m_j\rangle = \sum_{m_\ell m_s} |\ell, s; m_\ell, m_s\rangle\langle \ell, s; m_\ell, m_s|\ell, s; j, m_j\rangle$, where the scalar products are the expansion coefficients called Clebsch-Gordan coefficients. Reading them from the table gives (suppressing ℓ, s):

$$|j = \tfrac{5}{2}, m_j = +\tfrac{5}{2}\rangle = |m_\ell = +2, m_s = +\tfrac{1}{2}\rangle$$

$$|j = \tfrac{5}{2}, m_j = +\tfrac{3}{2}\rangle = \sqrt{\tfrac{1}{5}}|+2, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{4}{5}}|+1, +\tfrac{1}{2}\rangle \qquad |j = \tfrac{3}{2}, m_j = +\tfrac{3}{2}\rangle = \sqrt{\tfrac{4}{5}}|+2, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{5}}|+1, +\tfrac{1}{2}\rangle$$

$$|j = \tfrac{5}{2}, m_j = +\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{5}}|+1, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{5}}|0, +\tfrac{1}{2}\rangle \qquad |j = \tfrac{3}{2}, m_j = +\tfrac{1}{2}\rangle = \sqrt{\tfrac{3}{5}}|+1, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{5}}|0, +\tfrac{1}{2}\rangle$$

$$|j = \tfrac{5}{2}, m_j = -\tfrac{1}{2}\rangle = \sqrt{\tfrac{3}{5}}|0, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{5}}|-1, +\tfrac{1}{2}\rangle \qquad |j = \tfrac{3}{2}, m_j = -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{5}}|0, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{3}{5}}|-1, +\tfrac{1}{2}\rangle$$

$$|j = \tfrac{5}{2}, m_j = -\tfrac{3}{2}\rangle = \sqrt{\tfrac{4}{5}}|-1, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{5}}|-2, +\tfrac{1}{2}\rangle \qquad |j = \tfrac{3}{2}, m_j = -\tfrac{3}{2}\rangle = \sqrt{\tfrac{1}{5}}|-1, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{4}{5}}|-2, +\tfrac{1}{2}\rangle$$

$$|j = \tfrac{5}{2}, m_j = -\tfrac{5}{2}\rangle = |m_\ell = -2, m_s = -\tfrac{1}{2}\rangle$$

Here one should note the orthogonality between all states, in particular those on the same line due to the relative + and − signs. To derive all these states, one starts with the "extreme" state $|j = \frac{5}{2}, m_j = +\frac{5}{2}\rangle = 1\,|m_\ell = +2, m_s = +\frac{1}{2}\rangle$ with coefficient 1 (chosen real by convention) since only possibility is maximum m_ℓ and m_s. Then, one operates with ladder operators $J_- = L_- + S_-$ on respective sides giving $|j = \frac{5}{2}, m_j = +\frac{3}{2}\rangle = \ldots$ above. Iterating with the ladder operators gives the following states with $j = 5/2$. The state $|j = \frac{3}{2}, m_j = +\frac{3}{2}\rangle$ is then constructed to be orthogonal to $|j = \frac{5}{2}, m_j = +\frac{3}{2}\rangle$ and have sum of squared coefficients equal unity. The ladder operators are then iteratively applied to get the remaining $j = 3/2$ states.

Problem 4 (4 p.):
Find the eigenvectors and corresponding energy eigenvalues of a spin $s = 1$ system with hamiltonian $H = A\,S_z^2 + B(S_x^2 - S_y^2)$, where $A \gg B$ are constants. Explain/motivate your procedure and any approximations used.

Solution:
Since $A \gg B$ use perturbation theory with $H_1 = B(S_x^2 - S_y^2)$ as perturbation applied to the unperturbed $H_0 = A\,S_z^2$. Then, since $|s = 1, m = 0, \pm 1\rangle$ are eigenstates to S^2 and S_z they are eigenstates to H_0: $H_0|1, m\rangle = AS_z^2|1, m\rangle = A(m\hbar)^2|1, m\rangle$, i.e. energy eigenvalue 0 for $|1, 0\rangle$ (ground state) and $A\hbar^2$ for the two degenerate states $|1, \pm 1\rangle$ (first excited states).
Perturbation $H_1 = B(S_x^2 - S_y^2) = B(S_+^2 + S_-^2)/2$, since $S_\pm = S_x \pm iS_y$. The non-degenerate ground state has first order energy shift $\Delta_0^{(1)} = \frac{B}{2}\langle 1, 0|(S_+^2 + S_-^2)|1, 0\rangle = 0$, since $S_\pm^2$ attempts to shift the state up/down in m by two units to $m = \pm 2$ but there are no such states ($m = 0, \pm 1$ only). The state $|1, 0\rangle = 0$ is zeroth order approximation eigenvector to H.
For the degenerate states $|1, \pm 1\rangle$, perturbation theory gives the matrix equation

$$\frac{B}{2}\begin{pmatrix} \langle 1, +1|(S_+^2 + S_-^2)|1, +1\rangle & \langle 1, +1|(S_+^2 + S_-^2)|1, -1\rangle \\ \langle 1, -1|(S_+^2 + S_-^2)|1, +1\rangle & \langle 1, -1|(S_+^2 + S_-^2)|1, -1\rangle \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \Delta_\ell^{(1)}\begin{pmatrix} a \\ b \end{pmatrix}$$

in terms of the first order energy shift $\Delta_\ell^{(1)}$ and the zeroth order eigenstates $|\ell^{(0)}\rangle = |1,+1\rangle\langle 1,+1|\ell^{(0)}\rangle + |1,-1\rangle\langle 1,-1|\ell^{(0)}\rangle \doteq \begin{pmatrix} \langle 1,+1|\ell^{(0)}\rangle \\ \langle 1,-1|\ell^{(0)}\rangle \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$.
Repeated use of $J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$ in the matrix equation gives $\begin{pmatrix} -\Delta_\ell^{(1)} & B\hbar^2 \\ B\hbar^2 & -\Delta_\ell^{(1)} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$. The secular equation $det\begin{pmatrix} -\Delta_\ell^{(1)} & B\hbar^2 \\ B\hbar^2 & -\Delta_\ell^{(1)} \end{pmatrix} = 0$ then gives $\Delta_\ell^{(1)} = \pm B\hbar^2$, which inserted in the matrix equation gives $B\hbar^2\begin{pmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$ with solutions $a = \pm b$, respectively for the positive/negative energy shift. The normalisation condition $|a|^2 + |b|^2 = 1$ then gives $a = \pm b = \frac{1}{\sqrt{2}}$, i.e. $|\ell^{(0)}\rangle = \frac{1}{\sqrt{2}}\left(|1,+1\rangle \pm |1,-1\rangle\right)$.
Final result: the zeroth order eigenvectors to H are $|1,0\rangle$ and $\frac{1}{\sqrt{2}}\left(|1,+1\rangle \pm |1,-1\rangle\right)$ with energies 0 and $(A\pm B)\hbar^2$, respectively.

Problem 5 (5 p.):
Consider elastic scattering of particles with mass m and wave-vector $\vec{k}$ from the potential $V(\vec{x}) = -C\frac{\hbar^2}{2mk}\left(4\pi\delta(\vec{x})\frac{4k^2}{4k^2-\mu^2} + \frac{q^2}{r}e^{-\mu r}\right)$ where $q = |\vec{q}| = |\vec{k}-\vec{k}'|$ is the momentum transfer, $r = |\vec{x}|$, and μ, C are constants.
a) Show that the scattering amplitude in Born approximation is $f_k^{(1)}(\theta,\varphi) = \frac{C}{k}\left[\frac{4k^2}{4k^2-\mu^2} + \frac{q^2}{q^2+\mu^2}\right]$
b) Use this result to show that the partial wave amplitude for S-wave scattering in the Born approximation is $f_0^{(1)} = \frac{C}{k}\left[\frac{4k^2}{4k^2-\mu^2} + 1 - \frac{\mu^2}{4k^2}\ln\left(\frac{4k^2+\mu^2}{\mu^2}\right)\right]$
c) Apply to this result the unitarity condition, $Re\left\{|kf_0^{(1)}|\right\} < 1/2$ for $k^2 \gg \mu^2$, to derive a limit on the mass $M_H c^2$ (in units of GeV) of the Higgs boson! This is possible since the above describes (a contribution to) the elastic scattering $WW \to WW$ of W-bosons in electroweak theory, with m the reduced mass of the WW system, $\mu = M_H\hbar/c$, and $C = \sqrt{2}G_F M_H^2 c^4/\left(16\pi(\hbar c)^3\right)$ with $G_F/(\hbar c)^3 = 1.16639\cdot 10^{-5}$ GeV^{-2}.
<u>Hints</u>: $\int_0^\infty e^{-ax}\sin bx\, dx = b/(a^2+b^2)$, $\int_{-1}^1 P_l(\cos\theta)P_{l'}(\cos\theta)d(\cos\theta) = 2\delta_{ll'}/(2l+1)$, $P_0 = 1$

Solution:
(a) The scattering amplitude in Born-approximation is given by

$$f_k^{(1)}(\theta,\varphi) = -\frac{4\pi^2 m}{\hbar^2}\langle\vec{k}'|V|\vec{k}\rangle = -\frac{2m}{\hbar^2}\frac{1}{4\pi}\int d^3x' e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'}V(\vec{x}')$$

Inserting the two contributions to the potential separately gives:

$$f_k^{(1)}(\theta,\varphi)_A = \frac{2m}{\hbar^2}\frac{1}{4\pi}\int d^3x' e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'}C\frac{\hbar^2}{2mk}4\pi\delta(\vec{x}')\frac{4k^2}{4k^2-\mu^2} = \frac{C}{k}\frac{4k^2}{4k^2-\mu^2}$$

$$f_k^{(1)}(\theta,\varphi)_B = \frac{2m}{\hbar^2}\frac{1}{4\pi}\int d^3x' e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'}C\frac{\hbar^2}{2mk}\frac{q^2}{r'}e^{-\mu r'} =$$

[using polar coordinates with $\vec{q} = \vec{k}-\vec{k}'$ along the z'-axis giving $\vec{q}\cdot\vec{x}' = qr'\cos\theta'$ where θ' is the angle between $\vec{q}$ and $\vec{x}'$ (not to be confused with the scattering angle θ) one obtains]

$$= \frac{C}{k}\frac{1}{4\pi}\int_0^\infty dr' r'^2\int_{-1}^1 d(\cos\theta')\int_0^{2\pi} d\varphi' e^{iqr'\cos\theta'}\frac{q^2}{r'}e^{-\mu r'} = \frac{C}{k}\int_0^\infty dr' q\sin qr' e^{-\mu r'} = \frac{C}{k}\frac{q^2}{q^2+\mu^2}$$

Adding the two contributions gives the result

$$f_k^{(1)}(\theta,\varphi) = \frac{C}{k}\left[\frac{4k^2}{4k^2-\mu^2} + \frac{q^2}{q^2+\mu^2}\right]$$

where $q^2 = 2k^2(1-\cos\theta)$ is the momentum transfer for elastic scattering
(b) The partial wave expansion of a scattering amplitude is given by $f_k(\theta,\varphi) = \sum_l (2l+1) f_l P_l(\cos\theta)$. The partial wave amplitude for S-wave scattering $f_0^{(1)}$ in the Born approximation is then obtained by multiplying both sides with $P_0(\cos\theta) = 1$ and integrating over $\cos\theta$ exploiting the orthogonality of the Legendre polynomials, $\int_{-1}^{1} d(\cos\theta) f_k^{(1)}(\theta,\varphi) = \int_{-1}^{1} d(\cos\theta)\sum_l(2l+1)f_lP_l(\cos\theta) = 2f_0^{(1)}$, which gives (inserting $q^2 = 2k^2(1-\cos\theta)$)

$$\begin{aligned}
f_0^{(1)} &= \frac{1}{2}\int_{-1}^{1} d(\cos\theta) f_k^{(1)}(\theta,\varphi) = \frac{1}{2}\int_{-1}^{1} d(\cos\theta)\frac{C}{k}\left[\frac{4k^2}{4k^2-\mu^2}+\frac{q^2}{q^2+\mu^2}\right] = \\
&= \frac{C}{2k}\int_{-1}^{1} d(\cos\theta)\left[\frac{4k^2}{4k^2-\mu^2}+1-\frac{\mu^2}{q^2+\mu^2}\right] = \\
&= \frac{C}{k}\left[\frac{4k^2}{4k^2-\mu^2}+1-\frac{\mu^2}{2}\int_{-1}^{1} d(\cos\theta)\frac{1}{2k^2(1-\cos\theta)+\mu^2}\right] = \\
&= \frac{C}{k}\left[\frac{4k^2}{4k^2-\mu^2}+1-\frac{\mu^2}{4k^2}\ln\left(\frac{4k^2+\mu^2}{\mu^2}\right)\right]
\end{aligned}$$

(c) Taking the limit $k^2 \gg \mu^2$ gives $kf_0^{(1)} = 2C$ and the unitarity constraint $Re\left\{|kf_0^{(1)}|\right\} < 1/2$ becomes $C < 1/4$, where $C = \sqrt{2}G_F M_H^2 c^4/\left(16\pi(\hbar c)^3\right)$. In other words $M_H^2 c^4 < 2\sqrt{2}\pi(\hbar c)^3/G_F = 761818\,\mathrm{GeV}^2$ which gives the limit $M_H c^2 < 873$ GeV.

Problem 6 (3 p.):
A qubit is a quantum system whose Hilbert space is two-dimensional (e.g. the two polarization states of a photon). Let $|0\rangle, |1\rangle$ be a qubit basis. Now, consider four noninteracting qubits $1,\ldots,4$ prepared in the pure state $|\Gamma\rangle = |\Psi_-\rangle_{12}\otimes|\Psi_-\rangle_{34}$ with $|\Psi_-\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$ one of the four Bell states.
a) Characterize the entanglement in $|\Gamma\rangle$, i.e., state which of the qubit pairs are entangled and which are not.
b) Suppose qubits 2 and 3 are measured in the Bell basis and $|\Psi_-\rangle$ is obtained. What is the resulting state of qubits 1 and 4? Comment on any change of entanglement between 1 and 4. Is interaction a necessary requirement to create quantum entanglement?

Solution:
(a) Entangled pairs: (1,2) and (3,4). Product state pairs: (1,3), (1,4), (2,3), and (2,4).
(b) Let $\hat{P} \equiv \hat{1}_1 \otimes |\Psi_-\rangle_{23}\langle\Psi_-|_{23}\otimes\hat{1}_4$ be the projector corresponding to the outcome $|\Psi_-\rangle$ by measuring qubits 2 and 3. Projective measurement:

$$|\Gamma\rangle \rightarrow \frac{\hat{P}|\Gamma\rangle}{\sqrt{\langle\Gamma|\hat{P}|\Gamma\rangle}} = |\Psi_-\rangle_{23}\otimes|\Psi_-\rangle_{14}.$$

Thus, the qubits 1 and 4 have become (maximally) entangled although they have never interacted!
[This phenomenon is called *entanglement swapping*, see M. Żukowski *et al.*, Phys. Rev. Lett. **71**, 4287 (1993).]