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Exam 2010-12-16

1FA352: Quantum Mechanics 10 hp

2010 fall semester: Physics Master & Engineering Physics

Time:	14–19, i.e. 5 hours, Polacksbackens skrivsal
Allowed aids:	Physics Handbook, Mathematics Handbook, Tashenbuch der Mathematik, enclosed collection of formulae, calculator.
Instructions:	- write legible (skriv läsligt), define symbols, motivate equations etc. - write your personal exam code on each sheet - at most one problem per sheet - put your solutions in number order in this cover

Good luck !

Note: Results available on the course web page on December 22nd.

Personal exam code:

Number of sheets per problem:

1	2	3	4	5	6

Results: Sum: Grade:

1	2	3	4	5	6

1. A beam of unpolarised spin 1/2 particles goes through a series of Stern-Gerlach magnets: the first with field along $\hat{z}$ and a slit selecting spin up, the second with field along $\hat{n}$ having the angle θ to $\hat{z}$ in the xz plane and a slit selecting spin up, the third with field along $\hat{z}$ and a slit selecting spin down. What is the intensity of the final beam relative to the incoming beam? (4 p.)
   Hints: The operator $|S_n;+\rangle\langle S_n;+|$ projects out the state with spin up along $\hat{n}$.
   $S_x = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)$, $S_y = i\frac{\hbar}{2}\left(|-\rangle\langle+| - |+\rangle\langle-|\right)$, $S_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$.

2. For an arbitrary energy eigenstate $|n\rangle$ of a harmonic oscillator, calculate the expectation values of x^2 and p^2. What physical information can you obtain/calculate from this? (4 p.)

3. An electron is in a d-orbital ($\ell = 2$). Give, with motivations/explanations, all possible states $|jm\rangle$ of total angular momentum j of the electron, expressed in terms of its spin and orbital angular momentum. (4 p.)

4. Find the eigenvectors and corresponding energy eigenvalues of a spin $s = 1$ system with hamiltonian $H = A\,S_z^2 + B(S_x^2 - S_y^2)$, where $A \gg B$ are constants. Explain/motivate your procedure and any approximations used. (4 p.)

5. Consider elastic scattering of particles with mass m and wave-vector $\vec{k}$ from the potential $V(\vec{x}) = -C\frac{\hbar^2}{2mk}\left(4\pi\delta(\vec{x})\frac{4k^2}{4k^2-\mu^2} + \frac{q^2}{r}e^{-\mu r}\right)$ where $q = |\vec{q}| = |\vec{k} - \vec{k}'|$ is the momentum transfer, $r = |\vec{x}|$, and μ, C are constants.
   a) Show that the scattering amplitude in Born approximation is $f_k^{(1)}(\theta,\varphi) = \frac{C}{k}\left[\frac{4k^2}{4k^2-\mu^2} + \frac{q^2}{q^2+\mu^2}\right]$
   b) Use this result to show that the partial wave amplitude for S-wave scattering in the Born approximation is $f_0^{(1)} = \frac{C}{k}\left[\frac{4k^2}{4k^2-\mu^2} + 1 - \frac{\mu^2}{4k^2}\ln\left(\frac{4k^2+\mu^2}{\mu^2}\right)\right]$
   c) Apply to this result the unitarity condition, $Re\left\{|kf_0^{(1)}|\right\} < 1/2$ for $k^2 \gg \mu^2$, to derive a limit on the mass $M_H c^2$ (in units of GeV) of the Higgs boson! This is possible since the above describes (a contribution to) the elastic scattering $WW \to WW$ of W-bosons in electroweak theory, with m the reduced mass of the WW system, $\mu = M_H\hbar/c$, and $C = \sqrt{2}G_F M_H^2 c^4/\left(16\pi(\hbar c)^3\right)$ with $G_F/(\hbar c)^3 = 1.16639\cdot10^{-5}$ GeV^{-2}.
   (5 p.)
   Hints: $\int_0^\infty e^{-ax}\sin bx\,dx = b/(a^2+b^2)$, $\int_{-1}^1 P_l(\cos\theta)P_{l'}(\cos\theta)d(\cos\theta) = 2\delta_{ll'}/(2l+1)$, $P_0 = 1$

6. A qubit is a quantum system whose Hilbert space is two-dimensional (e.g. the two polarization states of a photon). Let $|0\rangle, |1\rangle$ be a qubit basis. Now, consider four noninteracting qubits $1,\ldots,4$ prepared in the pure state $|\Gamma\rangle = |\Psi_-\rangle_{12}\otimes|\Psi_-\rangle_{34}$ with $|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ one of the four Bell states.
   a) Characterize the entanglement in $|\Gamma\rangle$, i.e., state which of the qubit pairs are entangled and which are not.
   b) Suppose qubits 2 and 3 are measured in the Bell basis and $|\Psi_-\rangle$ is obtained. What is the resulting state of qubits 1 and 4? Comment on any change of entanglement between 1 and 4. Is interaction a necessary requirement to create quantum entanglement?
   (3p)

Collection of formulae

Angular momentum: $J_\pm = J_x \pm iJ_y$ $\quad J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$

Addition: $|j_1,j_2;j=j_1+j_2,m=j\rangle = |j_1,j_2;m_1=j_1,m_2=j_2\rangle$

$[x_i,F(\bar{p})] = i\hbar\frac{\partial F}{\partial p_i}$; $[p_i,G(\bar{x})] = -i\hbar\frac{\partial G}{\partial x_i}$

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ $\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Harmonic oscillator: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right)$ $\quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{ip}{m\omega}\right)$ $\quad [a,a^\dagger]=1$ $\quad N = a^\dagger a$

$a|n\rangle = \sqrt{n}|n-1\rangle$ $\quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

Time-independent perturbation theory:
Non-degenerate eigenvalue
$|n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)}-E_k^{(0)}} + \cdots$; $\Delta_n \equiv E_n - E_n^{(0)} = \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)}-E_k^{(0)}} + \cdots$

Degenerate eigenvalue
$|\ell\rangle = |\ell^{(0)}\rangle + \lambda|\ell^{(1)}\rangle + \cdots$ $\quad \Delta_\ell = \lambda\Delta_\ell^{(1)} + \lambda^2\Delta_\ell^{(2)} + \cdots$

$$\begin{pmatrix} V_{11} & V_{12} & \cdots \\ V_{21} & V_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix} = \Delta_\ell^{(1)}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix}$$

Time dependent perturbation theory:

$$\begin{aligned} c_n^{(0)}(t) &= \delta_{ni} \\ c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}V_{ni}(t')dt' \\ \omega_{ni} &= \frac{E_n-E_i}{\hbar} \end{aligned}$$

The Golden Rule: $w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n-E_i)$

The scattering amplitude: $f(\mathbf{k}',\mathbf{k}) = -\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\langle\mathbf{k}'|T|\mathbf{k}\rangle \overset{V=V(r)}{=} \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta)$

Perturbative expansion: $T = V + VG^+V + \ldots$

Partial wave amplitude: $f_l(k) = \frac{1}{2ik}(e^{2i\delta_l}-1) = \frac{1}{k}e^{i\delta_l}\sin\delta_l = \frac{1}{k\cot\delta_l - ik}$

Total elastic scattering cross section: $\sigma_{tot} = \int|f(\mathbf{k}',\mathbf{k})|^2 d\Omega \overset{V=V(r)}{=} 4\pi\sum_{l=0}^{\infty}(2l+1)|f_l(k)|^2$

Plane wave: $\langle\mathbf{x}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}$, $e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\hat{k}\cdot\hat{r})$

Clebsch-Gordan coefficients and spherical harmonics

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	…
		M	M	…
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

1/2×1/2

m_1	m_2	1, +1
+1/2	+1/2	1

m_1	m_2	1, 0	0, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1, −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

2×1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

1×1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

3/2×1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

2×1/2

m_1	m_2	5/2, +5/2
+2	1/2	1

m_1	m_2	5/2, 3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

Spherical Bessel and Neumann functions

$$j_l(kr) \overset{r\to\infty}{\to} \frac{1}{2ik}\left[\frac{e^{i(kr-l\pi/2)}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r}\right]$$

$$j_l(kr) \overset{kr\to 0}{\to} \frac{2^l l!}{(2l+1)!}(kr)^l \qquad j_0(kr) = \frac{sin(kr)}{kr}$$

$$n_l(kr) \overset{kr\to 0}{\to} -\frac{(2l)!}{2^l l!}\frac{1}{(kr)^{l+1}} \qquad n_0(kr) = -\frac{\cos(kr)}{kr}$$

Scattering length: $a \equiv a_0 = -\lim_{k\to 0}\dfrac{\tan\delta_0}{k}$

For inelastic scattering we have $d\sigma(0\to n) = w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]}/j_{in}$
where the transition rate is given by

$$w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]} = \frac{2\pi}{\hbar}\,|\langle\mathbf{k}',n|V|\mathbf{k},0\rangle|^2\,\left(\frac{L}{2\pi}\right)^3\left(\frac{mk'}{\hbar^2}\right)\,d\Omega$$

at box normalisation, $\langle\mathbf{x}|\mathbf{k},n\rangle = \frac{1}{L^{3/2}}\,e^{i\mathbf{k}\cdot\mathbf{x}}|n\rangle$.

$$\int d^3x\,\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{r} = \frac{4\pi}{q^2}$$