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Dept. of Physics and Astromony
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Examination 2009-10-20 in

Quantum Mechanics, advanced course

(Kvantmekanik fk, 5 poäng, 1TT173, F4T)

Engineering Physics (teknisk fysik)

Time:	8–13, i.e. 5 hours, Polacksbackens skrivsal
Allowed aids:	Physics Handbook, Mathematics Handbook, Tashenbuch der Mathematik, enclosed collection of formulae, calculator.
Instructions:	- write legible (skriv läsligt), define symbols, motivate equations etc. - write your name on each sheet, at most one problem per sheet - put your solutions in number order in this cover

Good luck !

Note:
Results available on www3.tsl.uu.se/thep/courses/QM/ on Friday October 30 (at the latest).

Name:

Personal number (personnummer):

Program (e.g. T, mat.nat.): Year of registration:

Number of sheets per problem:

1	2	3	4	5

Results: Sum: Grade:

1	2	3	4	5

1. Consider an observable A with orthonormal eigenvectors $|a'\rangle$, i.e. $A|a'\rangle = a'|a'\rangle$, and an arbitrary ket $|\alpha\rangle$ which can be expanded as $|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$.
(a) Show that $\sum_{a'} |a'\rangle\langle a'| = 1$ $\quad$ ($\Lambda_{a'} = |a'\rangle\langle a'|$ is the projection operator along $|a'\rangle$).
(b) Show that $\sum_{a'} |c_{a'}|^2 = 1$, if $|\alpha\rangle$ is normalized to unity.
(c) Show that the operator A can be written in the form $A = \sum_{a'} a'|a'\rangle\langle a'|$.
(d) Evaluate the expectation value $\langle A\rangle = \langle\alpha|A|\alpha\rangle$.
(4 p.)

2. (a) Define compatible and incompatible observables and explain the physical relevance of these concepts.
(b) For an arbitrary energy eigenstate $|n\rangle$ of a simple harmonic oscillator, calculate $\Delta x \Delta p$ (where $\Delta A = \sqrt{\langle (A - \langle A\rangle)^2\rangle} = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$) and show that it is minimal for the ground state. Comment on the physical meaning.
(4 p.)

3. Given a system of three spin 1/2 particles, express (with motivation) the total angular momentum state $|j, m\rangle$ in terms of $|m_1, m_2, m_3\rangle$ states of the three spins, i.e. $|+++\rangle, |++-\rangle, |+-+\rangle$ etc.

   Hint: Add first two spins to resulting intermediate states $|j', m'\rangle$, and then the third spin to obtain all states $|j, m\rangle$ for the total angular momentum.
(4 p.)

4. The Hamiltonian of a rigid diatomic molecule in a weak magnetic field in the xz-plane is $H = L^2/2I + BL_z + CL_x$, where I is the moment of inertia and $C/B = \tan\theta$ with θ the polar angle of the field. For $C \ll B$, find the approximate energy eigenvalues of H in second order perturbation theory (lowest non-vanishing order).

   Hint: Regard $H_0 = L^2/2I + BL_z$ as the unperturbed Hamiltonian and start from its energy eigenvalues and eigenstates.
(4 p.)

5. Calculate in the Born approximation the differential scattering cross section $d\sigma/dq$, where q is the momentum transfer $q = |\mathbf{k} - \mathbf{k}'|$, for scattering of particles with energy $E = \hbar^2 k^2/2m$ on the repulsive potential $V = V_0\, e^{-r/a}$.

   Hint: $\int_0^\infty dr\, r^p e^{-\alpha r} = \frac{p!}{\alpha^{p+1}}$ for $Re(\alpha) > 0$.
(4 p.)

Collection of formulae

Angular momentum: $J_\pm = J_x \pm iJ_y \qquad J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$

Addition: $|j_1,j_2;j=j_1+j_2,m=j\rangle = |j_1,j_2;m_1=j_1,m_2=j_2\rangle$

$[x_i,F(\bar{p})] = i\hbar\frac{\partial F}{\partial p_i}$; $[p_i,G(\bar{x})] = -i\hbar\frac{\partial G}{\partial x_i}$

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Harmonic oscillator: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \qquad [a,a^\dagger] = 1 \qquad N = a^\dagger a$

$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

Time-independent perturbation theory:
Non-degenerate eigenvalue
$|n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)}-E_k^{(0)}} + \cdots$; $\Delta_n \equiv E_n - E_n^{(0)} = \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)}-E_k^{(0)}} + \cdots$

Degenerate eigenvalue
$|\ell\rangle = |\ell^{(0)}\rangle + \lambda|\ell^{(1)}\rangle + \cdots \qquad \Delta_\ell = \lambda\Delta_\ell^{(1)} + \lambda^2\Delta_\ell^{(2)} + \cdots$

$$\begin{pmatrix} V_{11} & V_{12} & \cdots \\ V_{21} & V_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix} = \Delta_\ell^{(1)}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix}$$

Time dependent perturbation theory:

$$\begin{aligned} c_n^{(0)}(t) &= \delta_{ni} \\ c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = \frac{-i}{\hbar}\int_{t_0}^{t}e^{i\omega_{ni}t'}V_{ni}(t')dt' \\ \omega_{ni} &= \frac{E_n - E_i}{\hbar} \end{aligned}$$

The Golden Rule: $w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i)$

The scattering amplitude: $f(\mathbf{k}',\mathbf{k}) = -\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\langle\mathbf{k}'|T|\mathbf{k}\rangle \overset{V=V(r)}{=} \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta)$

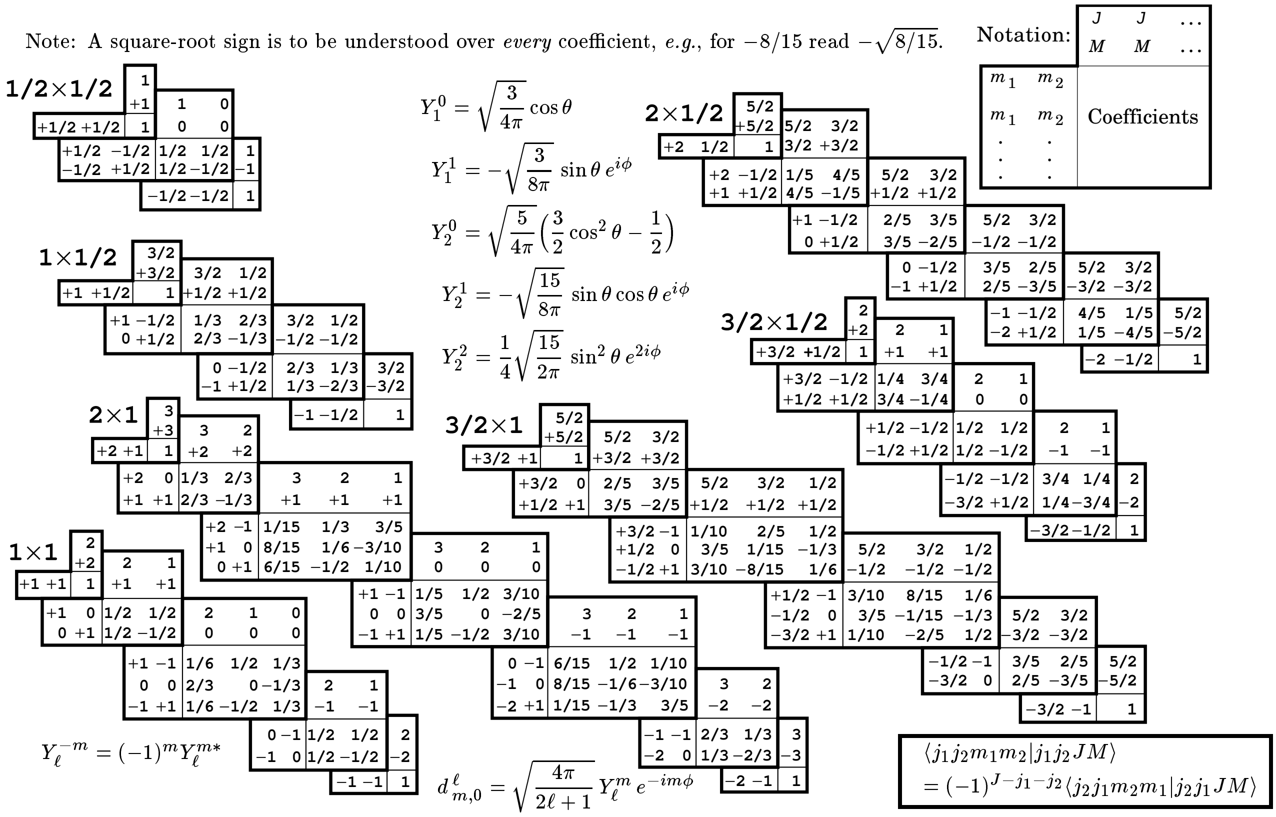
Perturbative expansion: $T = V + VG^+V + \ldots$

Partial wave amplitude: $f_l(k) = \frac{1}{2ik}(e^{2i\delta_l}-1) = \frac{1}{k}e^{i\delta_l}\sin\delta_l = \frac{1}{k\cot\delta_l - ik}$

Total elastic scattering cross section: $\sigma_{tot} = \int|f(\mathbf{k}',\mathbf{k})|^2d\Omega \overset{V=V(r)}{=} 4\pi\sum_{l=0}^{\infty}(2l+1)|f_l(k)|^2$

Plane wave: $\langle\mathbf{x}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}$, $e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty}(2l+1)i^lj_l(kr)P_l(\hat{k}\cdot\hat{r})$

Clebsch-Gordan coefficients and spherical harmonics



Spherical Bessel and Neumann functions

$$j_l(kr) \overset{r\to\infty}{\to} \frac{1}{2ik}\left[\frac{e^{i(kr-l\pi/2)}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r}\right]$$

$$j_l(kr) \overset{kr\to 0}{\to} \frac{2^l l!}{(2l+1)!}(kr)^l \qquad j_0(kr) = \frac{sin(kr)}{kr}$$

$$n_l(kr) \overset{kr\to 0}{\to} -\frac{(2l)!}{2^l l!}\frac{1}{(kr)^{l+1}} \qquad n_0(kr) = -\frac{\cos(kr)}{kr}$$

Scattering length: $a \equiv a_0 = -\lim_{k\to 0}\frac{\tan\delta_0}{k}$

For inelastic scattering we have $d\sigma(0\to n) = w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]}/j_{in}$
where the transition rate is given by

$$w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]} = \frac{2\pi}{\hbar}\,|\langle\mathbf{k}',n|V|\mathbf{k},0\rangle|^2\left(\frac{L}{2\pi}\right)^3\left(\frac{mk'}{\hbar^2}\right)d\Omega$$

at box normalisation, $\langle\mathbf{x}|\mathbf{k},n\rangle = \frac{1}{L^{3/2}}\,e^{i\mathbf{k}\cdot\mathbf{x}}|n\rangle$.

$$\int d^3x\,\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{r} = \frac{4\pi}{q^2}$$