

Problem 1 (4 p.):

A beam of atoms with total angular momentum $j = 1$ passes through a Stern-Gerlach magnet with field in direction \hat{n} in the xz plane at the angle θ to the z axis. The outgoing beam with $J_n = +\hbar$ is selected to pass another Stern-Gerlach magnet with field in the z direction and is then split into three beams. Determine the relative intensity of these beams.

Solution:

Magnet field in direction $\hat{n} = \sin\theta\hat{x} + \cos\theta\hat{z}$ implies $J_n = \sin\theta J_x + \cos\theta J_z$, where $J_x = (J_+ + J_-)/2$. Beam with spin $+\hbar$ along \hat{n} means satisfied eigenvalue equation $J_n|j_n, +1\rangle = +\hbar|j_n, +1\rangle$. The second Stern-Gerlach magnet along \hat{z} will project out eigenstates to J_z . Therefore express the state $|j_n, +1\rangle$ as linear combination of eigenstates to J_z , *i.e.* make the Ansatz $|j_n, +1\rangle = a|j_z, +1\rangle + b|j_z, 0\rangle + c|j_z, -1\rangle$, where $|a|^2 + |b|^2 + |c|^2 = 1$ for normalisation. Inserting this in left-hand side of the eigenvalue equation gives

$$\begin{aligned} J_n|j_n, +1\rangle &= \{\sin\theta(J_+ + J_-)/2 + \cos\theta J_z\} \{a|j_z, +1\rangle + b|j_z, 0\rangle + c|j_z, -1\rangle\} = \dots \\ &= \hbar \left(\frac{b}{\sqrt{2}} \sin\theta + a \cos\theta \right) |j_z, +1\rangle + \hbar \frac{a+c}{\sqrt{2}} \sin\theta |j_z, 0\rangle + \hbar \left(\frac{b}{\sqrt{2}} \sin\theta - c \cos\theta \right) |j_z, -1\rangle \end{aligned}$$

Equality with right-hand side of the eigenvalue equation implies equating the coefficients for $|j_z, +1\rangle$, $|j_z, 0\rangle$, and $|j_z, -1\rangle$, respectively. This gives the system of equations

$$a = \frac{b}{\sqrt{2}} \sin\theta + a \cos\theta, \quad b = \frac{a+c}{\sqrt{2}} \sin\theta, \quad c = \frac{b}{\sqrt{2}} \sin\theta - c \cos\theta$$

which together with the normalisation condition gives the solution $a = (1 + \cos\theta)/2$, $b = \frac{1}{\sqrt{2}} \sin\theta$, $c = (1 - \cos\theta)/2$. The relative intensities of the final beams with $m = 0, \pm 1$ along \hat{z} are then given by $|\langle j_z, m = 0, \pm 1 | j_n, 1 \rangle|^2$, *i.e.* by $|a|^2 = (1 + \cos\theta)^2/4$, $|b|^2 = \frac{1}{2} \sin^2\theta$ and $|c|^2 = (1 - \cos\theta)^2/4$, for $m = +1, 0, -1$ respectively. These relative intensities sum to unity for any θ and, as cross-check on the result being correct/reasonable, one observes that $(|a|^2, |b|^2, |c|^2)$ equals $(1, 0, 0)$ for $\theta = 0$, $(1/4, 1/2, 1/4)$ for $\theta = \pi/2$ and $(0, 0, 1)$ for $\theta = \pi$.

Problem 2 (4 p.):

- Specify the unitary time evolution operator $\mathcal{U}(t, t_0)$.
- Give the relation between an observable A in the Heisenberg and Schrödinger pictures.
- Derive the Heisenberg equation of motion $\frac{dA_H(t)}{dt} = \frac{1}{i\hbar} [A_H, H]$.

Solution:

(a) $\mathcal{U}(t, t_0) = e^{-i\frac{H}{\hbar}(t-t_0)}$, where H is the Hamiltonian operator, provides time evolution from t_0 to t of states in the Schrödinger picture ($|\alpha, t_0; t\rangle = \mathcal{U}(t, t_0)|\alpha, t_0\rangle$) and operators in the Heisenberg picture.

(b) $A_H(t) = \mathcal{U}^\dagger(t, t_0) A_S \mathcal{U}(t, t_0)$ obtained from equal expectation values in S- and H-picture, *i.e.* $\langle \alpha, t_0; t | A_S | \alpha, t_0; t \rangle = \langle \alpha, t_0 | \mathcal{U}^\dagger(t, t_0) A_S \mathcal{U}(t, t_0) | \alpha, t_0 \rangle = \langle \alpha, t_0 | A_H | \alpha, t_0 \rangle$

(c) Take $dA_H(t)/dt$ of expression in (b), using \mathcal{U} in (a), A_S time independent and $[\mathcal{U}, H] = 0$. Details in Sakurai page 83.

Problem 3 (4 p.):

Add the angular momenta $j_1 = 1$ and $j_2 = 1$ to form all possible states $|j, m\rangle$, of total angular momentum j , expressed as linear combinations of $|j_1 j_2; m_1 m_2\rangle$. Explain and motivate your procedure.

Solution:

The new basis states $|j_1 j_2; j m\rangle = \sum_{m_1 m_2} |j_1 j_2; m_1 m_2\rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2; j m\rangle$ expressed in the old direct product basis, where the scalar products are the Clebsch-Gordan coefficients (CG). Dropping $j_1 j_2$ which should be present in all kets and bras, the "extreme" state is $|j = 2, m = +2\rangle = |m_1 = +1, m_2 = +1\rangle$, *i.e.* this CG is chosen $=+1$ by the phase convention. Applying the ladder operator $J_- = J_{1-} + J_{2-}$ on this gives $J_- |j = 2, m = +2\rangle = (J_{1-} + J_{2-}) |m_1 = +1, m_2 = +1\rangle$ which results in

$$\begin{aligned} \sqrt{4}|j = 2, m = +1\rangle &= \sqrt{2}|m_1 = 0, m_2 = +1\rangle + \sqrt{2}|m_1 = +1, m_2 = 0\rangle \\ |j = 2, m = +1\rangle &= \frac{1}{\sqrt{2}}(|m_1 = 0, m_2 = +1\rangle + |m_1 = +1, m_2 = 0\rangle) \end{aligned}$$

Iterating with these ladder operators and applying orthonormality condition between states of different $j = 2, 1, 0$ gives

$$\begin{aligned} |j = 2, m = 2\rangle &= |m_1 = +1, m_2 = +1\rangle \\ |2, +1\rangle &= \sqrt{\frac{1}{2}}|0, +1\rangle + \sqrt{\frac{1}{2}}|+1, 0\rangle & |j = 1, m = +1\rangle &= \sqrt{\frac{1}{2}}|+1, 0\rangle - \sqrt{\frac{1}{2}}|0, +1\rangle \\ |2, 0\rangle &= \sqrt{\frac{1}{6}}|+1, -1\rangle + \sqrt{\frac{2}{3}}|0, 0\rangle + \sqrt{\frac{1}{6}}|-1, +1\rangle & |1, 0\rangle &= \sqrt{\frac{1}{2}}|+1, -1\rangle - \sqrt{\frac{1}{2}}|-1, +1\rangle \\ |2, -1\rangle &= \sqrt{\frac{1}{2}}|0, -1\rangle + \sqrt{\frac{1}{2}}|-1, 0\rangle & |1, -1\rangle &= \sqrt{\frac{1}{2}}|0, -1\rangle - \sqrt{\frac{1}{2}}|-1, 0\rangle \\ |j, -2\rangle &= |-1, -1\rangle \\ |j = 0, m = 0\rangle &= \sqrt{\frac{1}{3}}|+1, -1\rangle - \sqrt{\frac{1}{3}}|0, 0\rangle + \sqrt{\frac{1}{3}}|-1, +1\rangle \end{aligned}$$

Here, states on the left-hand side of "=" are $|j_1 j_2; j, m\rangle$ in the new basis and states on the right-hand side are $|j_1 j_2; m_1, m_2\rangle$ in the old direct product basis. The Clebsch-Gordan coefficients can also be directly read from the table, with explanation of which coefficient that should be placed where in the above solution.

Problem 4 (4 p.):

An isotropic harmonic oscillator in three dimensions has the Hamiltonian

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{m\omega^2}{2} (x^2 + y^2 + z^2).$$

- (a) What are the energies and eigenstates for the ground level and the first excited level?
 (b) A perturbation $V = \epsilon m\omega^2 xz$ is applied. For the states in (a), find the corresponding zeroth-order energy eigenstates and their energies to first order.

Solution:

(a) H_0 consists of three independent harmonic oscillators in the x, y, z -directions. The eigenstates are therefore $|n_x, n_y, n_z\rangle = |n_x\rangle|n_y\rangle|n_z\rangle$, with energy eigenvalues $E = \hbar\omega(n_x +$

$n_y + n_z + 3/2$). Thus, the ground level is $|0, 0, 0\rangle$ with $E = \hbar\omega\frac{3}{2}$ and the 3-fold degenerate first excited level $|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle$ with energy $E = \hbar\omega\frac{5}{2}$.

(b) Zeroth order ground state to $H = H_0 + V$ is $|0, 0, 0\rangle$ with $E_{000} = E_{000}^{(0)} + \Delta_{000}^{(0)} = \hbar\omega\frac{3}{2} + \epsilon m\omega^2 \langle 000|xz|000\rangle = \hbar\omega\frac{3}{2}$, since $\langle 000|xz|000\rangle = \langle 0|x|0\rangle\langle 0|0\rangle\langle 0|z|0\rangle = 0$ due to $x \sim a_x + a_x^\dagger$ giving $\langle 0|x|0\rangle = 0$ (and analogously for z).

For first excited states apply degenerate perturbation theory, giving

$$V \doteq \epsilon m\omega^2 \begin{pmatrix} \langle 100|xz|100\rangle & \langle 100|xz|010\rangle & \langle 100|xz|001\rangle \\ \langle 010|xz|100\rangle & \langle 010|xz|010\rangle & \langle 010|xz|001\rangle \\ \langle 001|xz|100\rangle & \langle 001|xz|010\rangle & \langle 001|xz|001\rangle \end{pmatrix}$$

With $x = \sqrt{\frac{\hbar}{2m\omega}}(a_x + a_x^\dagger)$, $\langle 1|x|0\rangle = \langle 0|x|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}$ and $\langle 0|x|0\rangle = \langle 1|x|1\rangle = 0$ and analogously $\langle 1|z|0\rangle = \langle 0|z|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}$ and $\langle 0|z|0\rangle = \langle 1|z|1\rangle = 0$, one obtains the matrix equation

$$\frac{\epsilon}{2}\hbar\omega \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \Delta^{(1)} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where $a = \langle 100|\ell^0\rangle$, $b = \langle 010|\ell^0\rangle$, $c = \langle 001|\ell^0\rangle$ give the first order perturbed states $|\ell^0\rangle = a|100\rangle + b|010\rangle + c|001\rangle$ expressed in the unperturbed states, satisfying the normalisation condition $|a|^2 + |b|^2 + |c|^2 = 1$. The secular equation $\det \begin{pmatrix} -\Delta & 0 & \mathcal{E} \\ 0 & -\Delta & 0 \\ \mathcal{E} & 0 & -\Delta \end{pmatrix} = 0$

gives the solution $\Delta = 0, \pm\mathcal{E}$, where $\mathcal{E} = \epsilon\hbar\omega/2$. Inserting these eigenvalues back into the matrix equation gives $a = 0, b = 1, c = 0$ for $\Delta = 0$, $a = 1/\sqrt{2}, b = 0, c = a$ for $\Delta = +\epsilon\hbar\omega/2$ and $a = 1/\sqrt{2}, b = 0, c = -a$ for $\Delta = -\epsilon\hbar\omega/2$, giving the resulting states

$$|010\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}(|100\rangle + |001\rangle) \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}}(|100\rangle - |001\rangle) \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

with respective energies $\hbar\omega\frac{5}{2}, \hbar\omega(\frac{5}{2} + \frac{\epsilon}{2}), \hbar\omega(\frac{5}{2} - \frac{\epsilon}{2})$.

The perturbation $V = \epsilon m\omega^2 xz$ leaves the y -oscillator unchanged, but mixes the x and z ones so that only the submatrix V_{xz} is nontrivial.

Problem 5 (4 p.):

(a) Show that for a spherically symmetric potential $V(r)$, the scattering amplitude $f(\vec{k}, \vec{k}')$ in first order Born approximation can be written

$$f^{(1)}(\theta, \phi) = -\frac{2m}{\hbar^2} \int_0^\infty dr' r'^2 \frac{\sin qr'}{qr'} V(r')$$

(b) Calculate the differential scattering cross-section $d\sigma/d\Omega$ in Born approximation for scattering of particles with momentum $\vec{p} = \hbar\vec{k}$ on the potential $V(r) = -V_0 e^{-r/R}$.

Hint: $\int_0^\infty dr r e^{-ar} = 1/a^2$ for $Re(a) > 0$.

Solution:

(a) For a weak potential, the perturbative expansion $T = V + VG^+V + \dots$ can be truncated after the first term giving the scattering amplitude $f(\vec{k}', \vec{k}) = -\frac{2m}{\hbar^2} \frac{(2\pi)^3}{4\pi} \langle \vec{k}' | V | \vec{k} \rangle$. Going to position space representation, by introducing completeness relations, one gets

$$\begin{aligned} f(\vec{k}', \vec{k}) &= -\frac{2m}{\hbar^2} \frac{(2\pi)^3}{4\pi} \int d\vec{x}' \int d\vec{x}'' \langle \vec{k}' | \vec{x}' \rangle \langle \vec{x}' | V | \vec{x}'' \rangle \langle \vec{x}'' | \vec{k} \rangle \\ &= -\frac{2m}{\hbar^2} \frac{(2\pi)^3}{4\pi} \int d\vec{x}' \int d\vec{x}'' \langle \vec{k}' | \vec{x}' \rangle V(\vec{x}'') \delta(\vec{x}' - \vec{x}'') \langle \vec{x}'' | \vec{k} \rangle \\ &= -\frac{2m}{\hbar^2} \frac{1}{4\pi} \int d\vec{x}' e^{i(\vec{k} - \vec{k}') \cdot \vec{x}'} V(\vec{x}') = -\frac{2m}{\hbar^2} \frac{1}{4\pi} \int d\vec{x}' e^{i\vec{q} \cdot \vec{x}'} V(\vec{x}') \end{aligned}$$

using $\langle \vec{x}' | \vec{k} \rangle = e^{i\vec{k} \cdot \vec{x}'} / (2\pi)^{3/2}$ and the momentum transfer $\vec{q} = \vec{k} - \vec{k}'$. For a spherically symmetric potential $V(r)$ one uses spherical coordinates, with the scalar product $\vec{q} \cdot \vec{x}' = qr' \cos \theta'$, giving

$$\begin{aligned} f(\theta, \phi) &= -\frac{2m}{\hbar^2} \frac{1}{4\pi} \int_0^{2\pi} d\phi' \int_{-1}^1 d \cos \theta' \int_0^\infty dr' r'^2 e^{iqr' \cos \theta'} V(r') \\ &= -\frac{2m}{\hbar^2} \frac{2\pi}{4\pi} \int_0^\infty dr' r'^2 \frac{e^{iqr'} - e^{-iqr'}}{iqr'} V(r') = -\frac{2m}{\hbar^2} \int_0^\infty dr' r'^2 \frac{\sin qr'}{qr'} V(r') \end{aligned}$$

which depends only on the magnitude $q = |\vec{k} - \vec{k}'| = \sqrt{k^2 - k'^2 - 2kk' \cos \theta} = 2k \sin \theta/2$ for elastic scattering when $|\vec{k}| = |\vec{k}'| = k$.

(b) For the potential $V(r) = -V_0 e^{-r/R}$, the scattering amplitude is

$$\begin{aligned} f(\theta, \phi) &= \frac{2mV_0}{\hbar^2} \int_0^\infty dr' r'^2 \frac{\sin qr'}{qr'} e^{-r'/R} = \frac{2mV_0}{\hbar^2 q} \int_0^\infty dr' r' \sin qr' e^{-r'/R} \\ &= \frac{2mV_0}{\hbar^2 q} \mathcal{I}m \int_0^\infty dr' r' e^{iqr'} e^{-r'/R} = \frac{2mV_0}{\hbar^2 q} \mathcal{I}m \int_0^\infty dr' r' e^{(iq-1/R)r'} \\ &= \frac{2mV_0}{\hbar^2 q} \mathcal{I}m \frac{1}{(1/R - iq)^2} = \frac{2mV_0}{\hbar^2} \frac{2R^3}{(1 + q^2 R^2)^2} \end{aligned}$$

where $\mathcal{I}m$ is the imaginary part and the hint has been used to evaluate the integral. The differential cross section is then

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(q)|^2 = \left(\frac{2mV_0}{\hbar^2} \right)^2 \frac{4R^6}{(1 + q^2 R^2)^4} = \left(\frac{2mV_0}{\hbar^2} \right)^2 \frac{4R^6}{(1 + 4k^2 R^2 \sin^2 \theta/2)^4}$$