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Examination 2008-10-21 in

Quantum Mechanics, advanced course

(Kvantmekanik fk, 5 poäng, 1TT173, F4T)

Engineering Physics (teknisk fysik)

Time:	9–14, i.e. 5 hours, at Gimogatan 4, sal 1
Allowed aids:	Physics Handbook, Mathematics Handbook, Tashenbuch der Mathematik, enclosed collection of formulae, calculator.
Instructions:	- write legible (skriv läsligt), define symbols, motivate equations etc. - write your name on each sheet, at most one problem per sheet - put your solutions in number order in this cover

Good luck !

Note:
Results available on www3.tsl.uu.se/thep/courses/QM/ on Friday October 31 (at the latest).

Name:

Personal number (personnummer):

Program (e.g. T, mat.nat.): Year of registration:

Number of sheets per problem:

1	2	3	4	5

Results: Sum: Grade:

1	2	3	4	5

1. A beam of atoms with total angular momentum $j = 1$ passes through a Stern-Gerlach magnet with field in direction $\hat{n}$ in the xz plane at the angle θ to the z axis. The outgoing beam with $J_n = +\hbar$ is selected to pass another Stern-Gerlach magnet with field in the z direction and is then split into three beams. Determine the relative intensity of these beams.

   (4 p.)

2. (a) Specify the unitary time evolution operator $\mathcal{U}(t, t_0)$.
   (b) Give the relation between an observable A in the Heisenberg and Schrödinger pictures.
   (c) Derive the Heisenberg equation of motion $\frac{dA_H(t)}{dt} = \frac{1}{i\hbar}[A_H, H]$.

   (4 p.)

3. Add the angular momenta $j_1 = 1$ and $j_2 = 1$ to form all possible states $|j, m\rangle$, of total angular momentum j, expressed as linear combinations of $|j_1 j_2; m_1 m_2\rangle$. Explain and motivate your procedure.

   (4 p.)

4. An isotropic harmonic oscillator in three dimensions has the Hamiltonian

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2 + z^2\right).$$

   (a) What are the energies and eigenstates for the ground level and the first excited level?
   (b) A perturbation $V = \epsilon m\omega^2 xz$ is applied. For the states in (a), find the corresponding zeroth-order energy eigenstates and their energies to first order.

   (4 p.)

5. (a) Show that for a spherically symmetric potential $V(r)$, the scattering amplitude $f(\vec{k}, \vec{k}')$ in first order Born approximation can be written

$$f^{(1)}(\theta, \phi) = -\frac{2m}{\hbar^2}\int_0^\infty dr'\, r'^2 \frac{\sin qr'}{qr'}\, V(r')$$

   (b) Calculate the differential scattering cross-section $d\sigma/d\Omega$ in Born approximation for scattering of particles with momentum $\vec{p} = \hbar\vec{k}$ on the potential $V(r) = -V_0 e^{-r/R}$.
   Hint: $\int_0^\infty dr\, r\, e^{-ar} = 1/a^2$ for $Re(a) > 0$.

   (4 p.)

Collection of formulae

Angular momentum: $J_\pm = J_x \pm iJ_y$ $\quad$ $J_\pm|j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m+1)}|j,m\pm 1\rangle$

Addition: $|j_1, j_2; j = j_1 + j_2, m = j\rangle = |j_1, j_2; m_1 = j_1, m_2 = j_2\rangle$

$[x_i, F(\bar{p})] = i\hbar\frac{\partial F}{\partial p_i}$; $[p_i, G(\bar{x})] = -i\hbar\frac{\partial G}{\partial x_i}$

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ $\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Harmonic oscillator: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$ $\quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$ $\quad [a, a^\dagger] = 1$ $\quad N = a^\dagger a$

$a|n\rangle = \sqrt{n}|n-1\rangle$ $\quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

Time-independent perturbation theory:
Non-degenerate eigenvalue
$|n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} + \cdots$; $\Delta_n \equiv E_n - E_n^{(0)} = \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \cdots$

Degenerate eigenvalue
$|\ell\rangle = |\ell^{(0)}\rangle + \lambda|\ell^{(1)}\rangle + \cdots$ $\quad \Delta_\ell = \lambda\Delta_\ell^{(1)} + \lambda^2\Delta_\ell^{(2)} + \cdots$

$$\begin{pmatrix} V_{11} & V_{12} & \cdots \\ V_{21} & V_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix} = \Delta_\ell^{(1)}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix}$$

Time dependent perturbation theory:

$$\begin{aligned} c_n^{(0)}(t) &= \delta_{ni} \\ c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}V_{ni}(t')dt' \\ \omega_{ni} &= \frac{E_n - E_i}{\hbar} \end{aligned}$$

The Golden Rule: $w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i)$

The scattering amplitude: $f(\mathbf{k}', \mathbf{k}) = -\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\langle\mathbf{k}'|T|\mathbf{k}\rangle \overset{V=V(r)}{=} \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta)$

Perturbative expansion: $T = V + VG^+V + \ldots$

Partial wave amplitude: $f_l(k) = \frac{1}{2ik}(e^{2i\delta_l} - 1) = \frac{1}{k}e^{i\delta_l}\sin\delta_l = \frac{1}{k\cot\delta_l - ik}$

Total elastic scattering cross section: $\sigma_{tot} = \int|f(\mathbf{k}', \mathbf{k})|^2 d\Omega \overset{V=V(r)}{=} 4\pi\sum_{l=0}^{\infty}(2l+1)|f_l(k)|^2$

Plane wave: $\langle\mathbf{x}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}$, $e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\hat{k}\cdot\hat{r})$

Clebsch-Gordan coefficients and spherical harmonics

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	…
		M	M	…
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\phi}$$

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	1/2	1

m_1	m_2	J=5/2, M=3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

Spherical Bessel and Neumann functions

$$j_l(kr) \overset{r\to\infty}{\to} \frac{1}{2ik}\left[\frac{e^{i(kr-l\pi/2)}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r}\right]$$

$$\begin{aligned} j_l(kr) &\overset{kr\to 0}{\to} \frac{2^l l!}{(2l+1)!}(kr)^l & \qquad j_0(kr) &= \frac{sin(kr)}{kr} \\ n_l(kr) &\overset{kr\to 0}{\to} -\frac{(2l)!}{2^l l!}\frac{1}{(kr)^{l+1}} & \qquad n_0(kr) &= -\frac{\cos(kr)}{kr} \end{aligned}$$

Scattering length: $a \equiv a_0 = -\lim_{k\to 0}\dfrac{\tan\delta_0}{k}$

For inelastic scattering we have $d\sigma(0\to n) = w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]}/j_{in}$
where the transition rate is given by

$$w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]} = \frac{2\pi}{\hbar}\,|\langle\mathbf{k}',n|V|\mathbf{k},0\rangle|^2\,\left(\frac{L}{2\pi}\right)^3\,\left(\frac{mk'}{\hbar^2}\right)\,d\Omega$$

at box normalisation, $\langle\mathbf{x}|\mathbf{k},n\rangle = \frac{1}{L^{3/2}}\,e^{i\mathbf{k}\cdot\mathbf{x}}|n\rangle$.

$$\int d^3x\,\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{r} = \frac{4\pi}{q^2}$$