Problem 1 (4 p.):
The eigenstates of two observables A and B provide two different complete orthonormal bases $\{|a^{(i)}\rangle\}$ and $\{|b^{(i)}\rangle\}$, respectively.
(a) Write the matrix representation $\overline{X}$ of an arbitrary operator X in the basis $\{|a^{(i)}\rangle\}$. Show that the matrix representation of the operator $X^\dagger$ is the Hermitian conjugated matrix.
(b) Show that $tr(XY) = tr(YX)$, where tr is the trace and X, Y are operators.
(c) Show that the trace of a matrix $\overline{X}$ is conserved in a unitary transformation between the bases $\{|a^{(i)}\rangle\}$ and $\{|b^{(i)}\rangle\}$, i.e. the trace is the same in both bases.
Hint: The trace is defined as sum of diagonal elements, i.e. $tr(X) = \sum_i \langle a^{(i)}|X|a^{(i)}\rangle$

Solution:
(a) By introducing completeness relations one gets $X = \sum_i \sum_j |a^{(i)}\rangle\langle a^{(i)}|X|a^{(j)}\rangle\langle a^{(j)}|$ and $X^\dagger = \sum_i \sum_j |a^{(i)}\rangle\langle a^{(i)}|X^\dagger|a^{(j)}\rangle\langle a^{(j)}| = \sum_i \sum_j |a^{(i)}\rangle\langle a^{(j)}|X|a^{(i)}\rangle^*\langle a^{(j)}|$. Here, $\langle ...|X|...\rangle$ can be seen as matrix elements when the indices i, j are varied in the "outer" ket $|a^{(i)}\rangle$ and the bra $\langle a^{(j)}|$, i.e.

$$X \doteq \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \dots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \qquad X^\dagger \doteq \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle^* & \langle a^{(2)}|X|a^{(1)}\rangle^* & \dots \\ \langle a^{(1)}|X|a^{(2)}\rangle^* & \langle a^{(2)}|X|a^{(2)}\rangle^* & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

This shows explicitly that $\overline{X^\dagger}$ is hermitian conjugated (transposed and complex conjugated) relative to the matrix $\overline{X}$. (See Sakurai p. 20.)
(b) $tr(XY) = \sum_i \langle a^{(i)}|XY|a^{(i)}\rangle = \sum_i \sum_j \langle a^{(i)}|X|a^{(j)}\rangle\langle a^{(j)}|Y|a^{(i)}\rangle = \sum_i \sum_j \langle a^{(j)}|Y|a^{(i)}\rangle\langle a^{(i)}|X|a^{(j)}\rangle = \sum_j \langle a^{(j)}|YX|a^{(j)}\rangle$ where a completeness relation (index j) have been inserted, giving two matrix elements (complex numbers) that commute, and finally a completeness relation (index i) has been removed.
(c) In $|a^{(i)}\rangle$ basis $tr(X) = \sum_i \langle a^{(i)}|X|a^{(i)}\rangle = \sum_i \sum_{j,k} \langle a^{(i)}|b^{(j)}\rangle\langle b^{(j)}|X|b^{(k)}\rangle\langle b^{(k)}|a^{(i)}\rangle = = \sum_i \sum_{j,k} \langle b^{(k)}|a^{(i)}\rangle\langle a^{(i)}|b^{(j)}\rangle\langle b^{(j)}|X|b^{(k)}\rangle = \sum_{j,k} \langle b^{(k)}|b^{(j)}\rangle\langle b^{(j)}|X|b^{(k)}\rangle = \sum_{j,k} \delta_{jk}\langle b^{(j)}|X|b^{(k)}\rangle = \sum_j \langle b^{(j)}|X|b^{(j)}\rangle = tr(X)$ which is the trace in $|b^{(j)}\rangle$ basis.

Problem 2 (4 p.):
A spin 1/2 particle has eigenstates $|+\rangle$ and $|-\rangle$ to the S_z operator. Consider the hamiltonian operator $H = A\left(|+\rangle\langle -| + |-\rangle\langle +|\right)$, where A is a constant.
(a) Find the energy eigenvalues and the corresponding eigenstates.
(b) For a particle having initially ($t = 0$) spin up (i.e. $+\hbar/2$ along $\hat{z}$), derive its state vector (in Schrödinger picture) for $t > 0$ and the probability that a measurement of S_z gives spin down ($-\hbar/2$).
(c) Give a physical interpretation of H and A.

Solution:
(a) The eigenstates $|+\rangle$, $|-\rangle$ of the observable S_z form a complete basis, but are not

eigenstates to the hamiltonian H, since

$$\begin{aligned} H|+\rangle &= A(|+\rangle\langle-| + |-\rangle\langle+|)|+\rangle = A|+\rangle\langle-|+\rangle + A|-\rangle\langle+|+\rangle = A|-\rangle\,, \\ H|-\rangle &= A(|+\rangle\langle-| + |-\rangle\langle+|)|-\rangle = A|+\rangle\langle-|-\rangle + A|-\rangle\langle+|-\rangle = A|+\rangle\,. \end{aligned}$$

The eigenstates $|E\rangle$ to H can be written $|E\rangle = a|+\rangle + b|-\rangle$, with $|a|^2 + |b|^2 = 1$. The eigenvalue equation $H|E\rangle = E|E\rangle$ then gives, for the right-hand and left-hand sides $H|E\rangle = A(|+\rangle\langle-| + |-\rangle\langle+|)(a|+\rangle + b|-\rangle) = A(a|-\rangle + b|+\rangle)$ and $E|E\rangle = E(a|+\rangle + b|-\rangle)$. Equating the coefficients for $|+\rangle$ and $|-\rangle$ gives $E = \pm A$ and $a = \pm b$, i.e. after normalisation

$$\begin{aligned} |E+\rangle &= \tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\,, \quad \text{with eigenvalue } A, \\ |E-\rangle &= \tfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\,, \quad \text{with eigenvalue } -A. \end{aligned}$$

Alternatively, this could have been obtained using the matrix representations $|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $H \doteq \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ and solving $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = E\begin{pmatrix} a \\ b \end{pmatrix}$.

(b) The initial state, at time $t = 0$, is $|\alpha(0)\rangle = |+\rangle$, while at a later time $t > 0$, the state of the system is given by the action of the time-evolution operator $|\alpha(t)\rangle = \mathcal{U}(t)|\alpha(0)\rangle = \mathrm{e}^{-\frac{i}{\hbar}Ht}|\alpha(0)\rangle$. Expressing the vectors $\{|+\rangle, |-\rangle\}$ in terms of the basis $\{|E+\rangle, |E-\rangle\}$, the state of the system becomes

$$\begin{aligned} |\alpha(t)\rangle &= \mathrm{e}^{-\frac{i}{\hbar}Ht}|+\rangle = \mathrm{e}^{-\frac{i}{\hbar}Ht}\tfrac{1}{\sqrt{2}}(|E+\rangle + |E-\rangle) = \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{i}{\hbar}Ht}|E+\rangle + \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{i}{\hbar}Ht}|E-\rangle = \\ &= \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{i}{\hbar}At}|E+\rangle + \tfrac{1}{\sqrt{2}}\mathrm{e}^{\frac{i}{\hbar}At}|E-\rangle = \tfrac{1}{2}\mathrm{e}^{-\frac{i}{\hbar}At}(|+\rangle + |-\rangle) + \tfrac{1}{2}\mathrm{e}^{\frac{i}{\hbar}At}(|+\rangle - |-\rangle) = \\ &= \cos(\tfrac{At}{\hbar})|+\rangle - i\sin(\tfrac{At}{\hbar})|-\rangle\,. \end{aligned}$$

The probability for spin down (state $|-\rangle$) is then $\mathcal{P}(t) = |\langle-|\alpha(t)\rangle|^2 = \sin^2\left(\frac{At}{\hbar}\right)$

(c) H is essentially S_x, or the Pauli matrix σ_x and can then describe the interaction with an external, static magnetic field $\vec{B}$ along the $\hat{x}$ direction, i.e. $H = -\left(\frac{e}{mc}\right)\vec{S}\cdot\vec{B} = -\left(\frac{eB}{mc}\right)S_x = \omega S_x$, where ω is the precession frequency, and $S_x = \frac{\hbar}{2}\sigma_x$ with σ_x the first Pauli matrix. Thus, the eigenvalues of the hamiltonian H are $\pm A = \pm\frac{\hbar\omega}{2}$.

Problem 3 (4 p.):
For a particle with spin $s = 1/2$ which is moving in a central force field one can choose as basis vectors either $|n\ell s; m_\ell m_s\rangle$ (the direct product basis) or $|n\ell s; jm\rangle$ where j, m are the quantum numbers for the total angular momentum $\vec{J} = \vec{L} + \vec{S}$. Consider a particle in the eigenstate to J^2 and J_z with $j = \ell + 1/2$, $m = \ell - 1/2$. Find the probability that a measurement of the z-component of the spin gives the value $S_z = \hbar/2$, i.e. the spin quantum number is $m_s = +1/2$, for the two cases $\ell = 1$ and $\ell = 2$, respectively.

Solution:
The new basis states are expanded in terms of the direct product basis states $|\ell s; jm\rangle = \sum_{m_\ell, m_s} |\ell s; m_\ell m_s\rangle\langle\ell s; m_\ell m_s|\ell s; jm\rangle$, where the scalar product (bra-ket) is a Clebsch-Gordan coefficients C_i that is non-zero only when $m = m_\ell + m_s$. This gives ($s = 1/2$ throughout)

$|\ell s; j = \ell + 1/2, m = \ell - 1/2\rangle = C_1|\ell s; m_\ell = \ell - 1, m_s = +1/2\rangle + C_2|\ell s; m_\ell = \ell, m_s = -1/2\rangle$
The probablility for spin up is then $|\langle \ldots m_s = +1/2|\ell s; j = \ell + 1/2, m = \ell - 1/2\rangle|^2$ where $\ldots$ indicate that the other quantum numbers can have any value sinze only S_z is measured. Thus, only the first state on the right-hand side gives a non-zero contribution, resulting the probability for $m_s = +1/2$ being $|C_1|^2$. Table of Clebsch-Gordan coefficients gives for $\ell = 1$, $s = 1/2 \Rightarrow |C_1|^2 = 2/3$ and for $\ell = 2$, $s = 1/2 \Rightarrow |C_1|^2 = 4/5$.

Alternative solution: The "extreme" state with maximum j and m can only be obtained by combining maximum m_ℓ and m_s, i.e. $|\ell s; j = \ell + 1/2, m = \ell + 1/2\rangle = 1|\ell s; m_\ell = \ell, m_s = 1/2\rangle$ where the Clebsch-Gordan coefficient $\langle \ell s; m_\ell = \ell, m_s = 1/2|\ell s; j = \ell + 1/2, m = \ell + 1/2\rangle = 1$ by the phase convention. Operating with $J_- = L_- + S_-$ on the two sides of this equation gives
LHS: $J_-|\ell s; j = \ell + 1/2, m = \ell + 1/2\rangle = \hbar\sqrt{(2\ell + 1)(1)}|\ell s; j = \ell + 1/2, m = \ell - 1/2\rangle$
RHS: $(L_- + S_-)|\ell s; m_\ell = \ell, m_s = 1/2\rangle = \hbar\sqrt{(\ell + \ell)(\ell - \ell + 1)}|\ell s; m_\ell = \ell - 1, m_s = 1/2\rangle + \hbar\sqrt{(s + 1/2)(s - 1/2 + 1)}|\ell s; m_\ell = \ell, m_s = -1/2\rangle$
and hence $|\ell s; j = \ell + 1/2, m = \ell - 1/2\rangle =$
$= \sqrt{\frac{2\ell}{2\ell+1}}|\ell s; m_\ell = \ell - 1, m_s = 1/2\rangle + \sqrt{\frac{1}{2\ell+1}}|\ell s; m_\ell = \ell, m_s = -1/2\rangle$
where the coefficient of the first term gives the probability for $m_s = +1/2$ to be $\frac{2\ell}{2\ell+1}$ giving 2/3 and 4/5 for $\ell = 1$ and $\ell = 2$, respectively.

Problem 4 (4 p.):
Consider a one-dimensional harmonic oscillator ($H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$) which initially ($t < 0$) is in the ground state $|0\rangle$. At $t = 0$ a perturbation of the form $V(x,t) = Ax^2exp(-t/\tau)$, is switched on. Calculate in first order time-dependent perturbation theory, the probability of finding the oscillator in any of the excited states $|n\rangle, n = 1, 2, 3, \ldots$, at time $t > 0$.

Solution:
Harmonic oscillator states $|n\rangle, n = 1, 2, 3, \ldots$ with energies $E_n = \hbar\omega(n + 1/2)$. Initial state is $|0\rangle$ and final states $|n\rangle$. This gives the transition amplitudes $c_n^{(0)}(t) = \delta_{n0}$ and $c_n^{(1)}(t) = \frac{-i}{\hbar}\int_0^t \langle n|V_I(t')|0\rangle dt' = \frac{-i}{\hbar}\int_0^t e^{i\omega_{n0}t'}V_{n0}(t')dt'$, where $\omega_{n0} = \frac{E_n - E_0}{\hbar} = n\omega$.
Here $V_{n0}(t') = \langle n|Ax^2e^{-t'/\tau}|0\rangle = Ae^{-t'/\tau}\langle n|x^2|0\rangle$. Using $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$, giving $x^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)(a + a^\dagger)$ and $a|0\rangle = 0$, $a|1\rangle = |0\rangle$, $a^\dagger|0\rangle = |1\rangle$, $a^\dagger|1\rangle = \sqrt{2}|2\rangle$, gives $x^2|0\rangle = \frac{\hbar}{2m\omega}\left(|0\rangle + \sqrt{2}|2\rangle\right)$ and hence $\langle n|x^2|0\rangle = \frac{\hbar}{2m\omega}\left(\delta_{n0} + \sqrt{2}\delta_{n2}\right)$ resulting in

$$c_2^{(1)}(t) = \frac{-i}{\hbar}\int_0^t e^{i2\omega t'}Ae^{-t'/\tau}\frac{\hbar\sqrt{2}}{2m\omega} = \frac{A\sqrt{2}}{i2m\omega}\frac{e^{2i\omega t - t/\tau} - 1}{2i\omega - 1/\tau}$$

The transition probability is then $P_{0\to n}(t) = 0$ for $n = 1, 3, 4, \ldots$ and $P_{0\to 2}(t) = |c_2^{(1)}(t)|^2 = \frac{A^2}{2m^2\omega^2(4\omega^2 + 1/\tau^2)}\left(1 + e^{-2t/\tau} - 2e^{-t/\tau}\cos 2\omega t\right)$

Problem 5 (4 p.):
Consider the elastic scattering of particles, with mass m and energy $E = \hbar^2 k^2/2m$, on the attractive spherically symmetric potential $V(r) = -V_0/r$ for $r < R$ and $V(r) = 0$ for $r > R$. Calculate the differential cross section $d\sigma/d\Omega$ in Born approximation. Comment on the case $R \to \infty$.

Solution:
The differential cross section $\frac{d\sigma}{d\Omega} = |f(\vec{k}', \vec{k})|^2$ with scattering amplitude $f(\vec{k}', \vec{k}) = -\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\langle \vec{k}'|T|\vec{k}\rangle$. Transition operator $T = V + VG^+V + \ldots \approx V$ in Born approximation gives

$$f(\vec{k}', \vec{k}) = -\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\langle \vec{k}'|V|\vec{k}\rangle = -\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\int d\vec{x}' \int d\vec{x}'' \langle \vec{k}'|\vec{x}'\rangle\langle \vec{x}'|V|\vec{x}''\rangle\langle \vec{x}''|\vec{k}\rangle =$$

$$= -\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\int d\vec{x}' \int d\vec{x}'' \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}'\cdot\vec{x}'} \langle \vec{x}'|V|\vec{x}''\rangle \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}''} =$$

$$= -\frac{2m}{\hbar^2}\frac{1}{4\pi}\int d\vec{x}' \int d\vec{x}'' e^{-i\vec{k}'\cdot\vec{x}'} V(\vec{x}'')\delta(\vec{x}' - \vec{x}'') e^{i\vec{k}\cdot\vec{x}''} = -\frac{2m}{\hbar^2}\frac{1}{4\pi}\int d\vec{x}' e^{i\vec{q}\cdot\vec{x}'} V(\vec{x}')$$

Using polar coordinates, with $\vec{q} = \vec{k} - \vec{k}'$ along $\hat{z}'$, and inserting the potential gives

$$f_k^{(1)}(\theta, \varphi) = \frac{2m}{\hbar^2}\frac{V_0}{4\pi}\int_0^R dr' r'^2 \int_{-1}^1 d(\cos\theta') \int_0^{2\pi} d\varphi' e^{iqr'\cos\theta'} \frac{1}{r'} = \frac{2m}{\hbar^2}\frac{V_0}{4\pi} 2\pi \int_0^R dr' r'^2 \frac{e^{iqr'} - e^{-iqr'}}{iqr'}\frac{1}{r'}$$

$$= \frac{mV_0}{\hbar^2}\int_0^R dr' r'^2 \frac{2\sin qr'}{qr'}\frac{1}{r'} = \frac{2mV_0}{\hbar^2}\frac{1}{q}\int_0^R dr' \sin qr' = \frac{2mV_0}{\hbar^2} \cdot \frac{1 - \cos qR}{q^2}$$

where the momentum transfer $q = |\vec{q}| = \sqrt{k^2 + k'^2 - 2kk'\cos\theta} = 2k\sin\theta/2$ for elastic scattering having $|\vec{k}'| = |\vec{k}| = k$. The differential cross-section in Born approximation is then

$$\frac{d\sigma^{(1)}}{d\Omega} = |f_k^{(1)}(\theta, \varphi)|^2 = \frac{m^2 V_0^2}{4\hbar^4}\frac{1}{k^4 \sin^4\frac{\theta}{2}}\left(1 - \cos qR\right)^2$$

For $R \to \infty$ the potential becomes the Coulomb potential. However, the above scattering amplitude (and thereby the cross section) is then not well defined, which originates from the above integration over r'. The potential need to be more strongly suppressed with increasing r' in order to get a well-behaved, converged integral. A potential $V(r) = -\frac{V_0}{r^p}$ with $p > 1$ is sufficient. To get exactly the Coloumb potential, one can insert an exponential suppression, as in the Yukawa potential $V(r) = -V_0\frac{e^{-r/R}}{r/R}$, to obtain a solvable integral and thereafter let $R \to \infty$ in a proper way to obtain the correct Coulomb scattering cross section (as discussed in a lecture).

From the above results one can, however, see that the result is almost the Coulomb scattering cross section, with its characteristic $1/\sin^4\frac{\theta}{2}$ behavior. The extra factor $1 - \cos qR$ is limited to the range $[0, 2]$ and does not alter the Coulomb cross section very much. For very large R, $\cos qR$ oscillates very fast giving a mean or typical value of $\cos qR = 0$, indicating the Coulomb cross section.