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Examination 2007-10-19 in
Quantum Mechanics, advanced course
(Kvantmekanik fk, 5 poäng, 1TT173, F4T)
Engineering Physics (teknisk fysik)

Time:	14–19, i.e. 5 hours, at Polacksbacken
Allowed aids:	Physics Handbook, Mathematics Handbook, Tashenbuch der Mathematik, enclosed collection of formulae, calculator.
Instructions:	- write legible (skriv läsligt), define symbols, motivate equations etc. - write your name on each sheet, at most one problem per sheet - put your solutions in number order in this cover

Good luck !

Note:
Results available on www3.tsl.uu.se/thep/courses/QM/ on Tuesday October 30 (at the latest).

Name:

Personal number (personnummer):

Program (e.g. T, mat.nat.): Year of registration:

Number of sheets per problem:

1	2	3	4	5

Results: Sum: Grade:

1	2	3	4	5

1. The eigenstates of two observables A and B provide two different complete orthonormal bases $\{|a^{(i)}\rangle\}$ and $\{|b^{(i)}\rangle\}$, respectively.
(a) Write the matrix representation $\overline{X}$ of an arbitrary operator X in the basis $\{|a^{(i)}\rangle\}$. Show that the matrix representation of the operator $X^\dagger$ is the Hermitian conjugated matrix.
(b) Show that $tr(XY) = tr(YX)$, where tr is the trace and X, Y are operators.
(c) Show that the trace of a matrix $\overline{X}$ is conserved in a unitary transformation between the bases $\{|a^{(i)}\rangle\}$ and $\{|b^{(i)}\rangle\}$, i.e. the trace is the same in both bases.
<u>Hint</u>: The trace is defined as sum of diagonal elements, i.e. $tr(X) = \sum_i \langle a^{(i)}|X|a^{(i)}\rangle$
(4 p.)

2. A spin 1/2 particle has eigenstates $|+\rangle$ and $|-\rangle$ to the S_z operator. Consider the hamiltonian operator $H = A\,(|+\rangle\langle-| \,+\, |-\rangle\langle+|)$, where A is a constant.
(a) Find the energy eigenvalues and the corresponding eigenstates.
(b) For a particle having initially ($t = 0$) spin up (i.e. $+\hbar/2$ along $\hat{z}$), derive its state vector (in Schrödinger picture) for $t > 0$ and the probability that a measurement of S_z gives spin down ($-\hbar/2$).
(c) Give a physical interpretation of H and A.
(4 p.)

3. For a particle with spin $s = 1/2$ which is moving in a central force field one can choose as basis vectors either $|n\ell s; m_\ell m_s\rangle$ (the direct product basis) or $|n\ell s; jm\rangle$ where j, m are the quantum numbers for the total angular momentum $\vec{J} = \vec{L} + \vec{S}$. Consider a particle in the eigenstate to J^2 and J_z with $j = \ell + 1/2$, $m = \ell - 1/2$. Find the probability that a measurement of the z-component of the spin gives the value $S_z = \hbar/2$, i.e. the spin quantum number is $m_s = +1/2$, for the two cases $\ell = 1$ and $\ell = 2$, respectively.
(4 p.)

4. Consider a one-dimensional harmonic oscillator ($H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$) which initially ($t < 0$) is in the ground state $|0\rangle$. At $t = 0$ a perturbation of the form $V(x,t) = Ax^2 exp(-t/\tau)$, is switched on. Calculate in first order time-dependent perturbation theory, the probability of finding the oscillator in any of the excited states $|n\rangle, n = 1, 2, 3, ...,$ at time $t > 0$.
(4 p.)

5. Consider the elastic scattering of particles, with mass m and energy $E = \hbar^2 k^2/2m$, on the attractive spherically symmetric potential

$$V(r) = \begin{cases} -V_0/r & \text{for } r < R, \\ 0 & \text{for } r > R, \end{cases}$$

Calculate the differential cross section $d\sigma/d\Omega$ in Born approximation. Comment on the case $R \to \infty$.
(4 p.)

Collection of formulae

Angular momentum: $J_\pm = J_x \pm iJ_y \qquad J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$

Addition: $|j_1,j_2;j=j_1+j_2,m=j\rangle = |j_1,j_2;m_1=j_1,m_2=j_2\rangle$

$[x_i,F(\bar p)] = i\hbar\frac{\partial F}{\partial p_i}$; $[p_i,G(\bar x)] = -i\hbar\frac{\partial G}{\partial x_i}$

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Harmonic oscillator: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \qquad [a,a^\dagger] = 1 \qquad N = a^\dagger a$

$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

Time-independent perturbation theory:
Non-degenerate eigenvalue
$|n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)}-E_k^{(0)}} + \cdots$; $\Delta_n \equiv E_n - E_n^{(0)} = \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)}-E_k^{(0)}} + \cdots$

Degenerate eigenvalue
$|\ell\rangle = |\ell^{(0)}\rangle + \lambda|\ell^{(1)}\rangle + \cdots \qquad \Delta_\ell = \lambda\Delta_\ell^{(1)} + \lambda^2\Delta_\ell^{(2)} + \cdots$

$$\begin{pmatrix} V_{11} & V_{12} & \cdots \\ V_{21} & V_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix} = \Delta_\ell^{(1)}\begin{pmatrix} \langle 1^{(0)}|\ell^{(0)}\rangle \\ \langle 2^{(0)}|\ell^{(0)}\rangle \\ \vdots \end{pmatrix}$$

Time dependent perturbation theory:

$$\begin{aligned} c_n^{(0)}(t) &= \delta_{ni} \\ c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = \frac{-i}{\hbar}\int_{t_0}^{t}e^{i\omega_{ni}t'}V_{ni}(t')dt' \\ \omega_{ni} &= \frac{E_n - E_i}{\hbar} \end{aligned}$$

The Golden Rule: $w_{i\to n} = \frac{2\pi}{\hbar}\,|V_{ni}|^2\delta(E_n - E_i)$

The scattering amplitude: $f(\mathbf{k}',\mathbf{k}) = -\,\frac{2m}{\hbar^2}\frac{(2\pi)^3}{4\pi}\langle\mathbf{k}'|T|\mathbf{k}\rangle \overset{V=V(r)}{=} \sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta)$

Perturbative expansion: $T = V + VG^+V + \ldots$

Partial wave amplitude: $f_l(k) = \frac{1}{2ik}(e^{2i\delta_l}-1) = \frac{1}{k}e^{i\delta_l}\sin\delta_l = \frac{1}{k\cot\delta_l - ik}$

Total elastic scattering cross section: $\sigma_{tot} = \int|f(\mathbf{k}',\mathbf{k})|^2d\Omega \overset{V=V(r)}{=} 4\pi\sum_{l=0}^{\infty}(2l+1)|f_l(k)|^2$

Plane wave: $\langle\mathbf{x}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}$, $e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\hat{k}\cdot\hat{r})$

Clebsch-Gordan coefficients and spherical harmonics

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	…
	M	M	…
m_1 m_2	Coefficients		
m_1 m_2			
⋮ ⋮			

1/2×1/2

	1
	+1
+1/2 +1/2	1

	1	0
	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	1
	−1
−1/2 −1/2	1

1×1/2

	3/2
	+3/2
+1 +1/2	1

	3/2	1/2
	+1/2	+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

	3/2	1/2
	−1/2	−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

	3/2
	−3/2
−1 −1/2	1

2×1

	3
	+3
+2 +1	1

	3	2
	+2	+2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

	3	2	1
	+1	+1	+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	6/15	−1/2	1/10

	3	2	1
	0	0	0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

	3	2	1
	−1	−1	−1
0 −1	6/15	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

	3	2
	−2	−2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

	3
	−3
−2 −1	1

1×1

	2
	+2
+1 +1	1

	2	1
	+1	+1
+1 0	1/2	1/2
0 +1	1/2	−1/2

	2	1	0
	0	0	0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

	2	1
	−1	−1
0 −1	1/2	1/2
−1 0	1/2	−1/2

	2
	−2
−1 −1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\phi}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

2×1/2

	5/2
	+5/2
+2 1/2	1

	5/2	3/2
	3/2	+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2	3/2
	+1/2	+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2	3/2
	−1/2	−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2	3/2
	−3/2	−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2
	−5/2
−2 −1/2	1

3/2×1/2

	2
	+2
+3/2 +1/2	1

	2	1
	+1	+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

	2	1
	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	2	1
	−1	−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

	2
	−2
−3/2 −1/2	1

3/2×1

	5/2
	+5/2
+3/2 +1	1

	5/2	3/2
	+3/2	+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

	5/2	3/2	1/2
	+1/2	+1/2	+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

	5/2	3/2	1/2
	−1/2	−1/2	−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

	5/2	3/2
	−3/2	−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

	5/2
	−5/2
−3/2 −1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

Spherical Bessel and Neumann functions

$$j_l(kr) \overset{r\to\infty}{\to} \frac{1}{2ik}\left[\frac{e^{i(kr-l\pi/2)}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r}\right]$$

$$\begin{array}{llll} j_l(kr) \overset{kr\to 0}{\to} \frac{2^l l!}{(2l+1)!}(kr)^l & & j_0(kr) & = \frac{sin(kr)}{kr} \\ n_l(kr) \overset{kr\to 0}{\to} -\frac{(2l)!}{2^l l!}\frac{1}{(kr)^{l+1}} & & n_0(kr) & = -\frac{\cos(kr)}{kr} \end{array}$$

Scattering length: $a \equiv a_0 = -\lim\limits_{k\to 0}\dfrac{\tan\delta_0}{k}$

For inelastic scattering we have $d\sigma(0\to n) = w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]}/j_{in}$
where the transition rate is given by

$$w_{\mathbf{k},0\to[\mathbf{k}'\in d\Omega,n]} = \frac{2\pi}{\hbar}\,|\langle\mathbf{k}',n|V|\mathbf{k},0\rangle|^2\left(\frac{L}{2\pi}\right)^3\left(\frac{mk'}{\hbar^2}\right)d\Omega$$

at box normalisation, $\langle\mathbf{x}|\mathbf{k},n\rangle = \frac{1}{L^{3/2}}\,e^{i\mathbf{k}\cdot\mathbf{x}}|n\rangle$.

$$\int d^3x\,\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{r} = \frac{4\pi}{q^2}$$