

Problem 1 (4 p.):

- (a) Define the concept of unitary operator and show that the operator $e^{ipa/\hbar}$, where p is the momentum operator in one dimension, is unitary.
- (b) Show that $e^{ipa/\hbar}|x'\rangle$ is an eigenstate to the position operator. What is the eigenvalue? (Hint: Consider $[x, e^{ipa/\hbar}]$.)

Solution:

- (a) An operator U is unitary if it satisfies $U^\dagger U = UU^\dagger = 1$. With $U = e^{ipa/\hbar}$, then $U^\dagger = e^{-ipa/\hbar}$ and, therefore, $U^\dagger U = e^{-ipa/\hbar} e^{ipa/\hbar} = 1 = e^{ipa/\hbar} e^{-ipa/\hbar} = UU^\dagger$.
- (b) With the position operator $x|x'\rangle = x'|x'\rangle$, we should demonstrate that $x e^{ipa/\hbar}|x'\rangle = C e^{ipa/\hbar}|x'\rangle$ where C is the eigenvalue:
 $x e^{ipa/\hbar}|x'\rangle = \{[x, e^{ipa/\hbar}] + e^{ipa/\hbar} x\}|x'\rangle = (\dots \text{using } [x_i, F(\vec{p})] = i\hbar \frac{\partial F}{\partial p_i} \text{ from the "collection of formulae" } \dots) = i\hbar \frac{\partial e^{ipa/\hbar}}{\partial p} |x'\rangle + e^{ipa/\hbar} x|x'\rangle = i\hbar \frac{ia}{\hbar} e^{ipa/\hbar} |x'\rangle + e^{ipa/\hbar} x|x'\rangle = (-a+x')e^{ipa/\hbar}|x'\rangle$,
 i.e. eigenstate of x with eigenvalue $x' - a$. This means a translation by a , which is in accordance with $e^{ipa/\hbar}$ being the translation operator.

Problem 2 (4 p.):

A one-dimensional harmonic oscillator is at time $t = 0$ in the state $|\alpha, t = 0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, where $|n\rangle$ are the energy-eigenstates with energy eigenvalues $E_n = \hbar\omega(n + 1/2)$.

- (a) Give the state $|\alpha, t\rangle$ at a later time t .
- (b) For this state, determine the expectation value of the kinetic energy $E_k = p^2/2m$ as a function of time.

Solution:

- (a) Applying the time evolution operator $e^{-iHt/\hbar}$, and using $e^{-iHt/\hbar}|n\rangle = e^{-i\omega(n+1/2)t}|n\rangle$, one gets $|\alpha, t\rangle = e^{-iHt/\hbar}|\alpha, t = 0\rangle = e^{-iHt/\hbar}(|0\rangle + |1\rangle)/\sqrt{2} = (e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle)/\sqrt{2}$ as the state vector for $t > 0$ (in the Schrödinger picture).
- (b) With the momentum operator $p = i\sqrt{\hbar m\omega/2}(a^\dagger - a)$, the kinetic energy operator is $E_k = \frac{p^2}{2m} = \frac{\hbar\omega}{4}(a^\dagger a - a^\dagger a^\dagger - aa + aa^\dagger)$. The expectation value is then $\langle E_k \rangle = \langle \alpha, t | E_k | \alpha, t \rangle = \frac{1}{\sqrt{2}}(e^{i\omega t/2}\langle 0 | + e^{3i\omega t/2}\langle 1 |) \frac{\hbar\omega}{4}(a^\dagger a - a^\dagger a^\dagger - aa + aa^\dagger) \frac{1}{\sqrt{2}}(e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle)$. Repeated use of $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, then gives $\langle E_k \rangle = \frac{\hbar\omega}{8}(e^{i\omega t/2}\langle 0 | + e^{3i\omega t/2}\langle 1 |)(e^{-i\omega t/2}|0\rangle + 3e^{-3i\omega t/2}|1\rangle) = \frac{\hbar\omega}{2}$. Note that this does not depend on time t and is the average of the kinetic energies of the two states $|0\rangle$ and $|1\rangle$.

Problem 3 (4 p.):

A physical system contains two particles with angular momenta \vec{J}_1 and \vec{J}_2 .

(a) Give the relation between the two bases which is given by the eigenvectors to the internally commuting operators $\vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}$ (direct product basis) and the operators $\vec{J}_1^2, \vec{J}_2^2, \vec{J}^2, J_z$ where $\vec{J} = \vec{J}_1 + \vec{J}_2$.

(b) Show that the Clebsch-Gordan coefficients are non-zero only if $m = m_1 + m_2$ where $m\hbar, m_1\hbar, m_2\hbar$ are the eigenvalues to J_z, J_{1z}, J_{2z} .

(c) Let the two particles have angular momenta 1 and 1/2, respectively, which are added to a total angular momentum j . Give (with motivation) all resulting states $|j, m\rangle$ expressed in $|j_1 = 1, m_1\rangle$ and $|j_2 = 1/2, m_2\rangle$.

Solution:

(a) The direct product basis kets $|j_1, j_2; m_1, m_2\rangle$ are eigenstates of the commuting operators $\vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}$, and the new basis kets $|j_1, j_2; j, m\rangle$ are eigenstates of $\vec{J}_1^2, \vec{J}_2^2, \vec{J}^2, J_z$. The new basis states are obtained in terms of the old basis states by introducing the completeness relation, giving the relation $|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} |j_1, j_2; m_1, m_2\rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle$, where the scalar products are the Clebsch-Gordan coefficients.

(b) $\vec{J} = \vec{J}_1 + \vec{J}_2 \Rightarrow J_z = J_{1z} + J_{2z} \Rightarrow J_z - J_{1z} - J_{2z} = 0$. Multiplying this last equation from the left/right with a bra/ket of the old/new basis gives $\langle j_1, j_2; m_1, m_2 | (J_z - J_{1z} - J_{2z}) | j_1, j_2; j, m\rangle = 0$ and with the (hermitean) operators acting to left/right one obtains $\hbar(m - m_1 - m_2) \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle = 0$, which shows that the Clebsch-Gordan coefficient can be non-zero only if $m = m_1 + m_2$.

(c) Adding $j_1 = 1$ and $j_2 = 1/2$ gives possible values $j = 1/2$ and $3/2$ (since $|j_1 - j_2| \leq j \leq j_1 + j_2$), with $m = -j \dots j$. Using the relation in (a) and taking the Clebsch-Gordan coefficients from the table gives:

$$\begin{aligned} |j = \frac{3}{2}, m = +\frac{3}{2}\rangle &= |m_1 = +1, m_2 = +\frac{1}{2}\rangle \\ |j = \frac{3}{2}, m = +\frac{1}{2}\rangle &= \sqrt{\frac{2}{3}}|m_1 = 0, m_2 = +\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|m_1 = +1, m_2 = -\frac{1}{2}\rangle \\ |j = \frac{3}{2}, m = -\frac{1}{2}\rangle &= \sqrt{\frac{1}{3}}|m_1 = -1, m_2 = +\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|m_1 = 0, m_2 = -\frac{1}{2}\rangle \\ |j = \frac{3}{2}, m = -\frac{3}{2}\rangle &= |m_1 = -1, m_2 = -\frac{1}{2}\rangle \end{aligned}$$

$$\begin{aligned} |j = \frac{1}{2}, m = +\frac{1}{2}\rangle &= -\sqrt{\frac{1}{3}}|m_1 = 0, m_2 = +\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|m_1 = +1, m_2 = -\frac{1}{2}\rangle \\ |j = \frac{1}{2}, m = -\frac{1}{2}\rangle &= -\sqrt{\frac{2}{3}}|m_1 = -1, m_2 = +\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|m_1 = 0, m_2 = -\frac{1}{2}\rangle \end{aligned}$$

Alternatively, one can derive these results using ladder operators. Start with the 'extreme' case $|j = \frac{3}{2}, m = +\frac{3}{2}\rangle = 1|m_1 = +1, m_2 = +\frac{1}{2}\rangle$ with coefficient 1 (chosen real by convention) since only possibility is maximum m_1 and m_2 , and then apply $J_- = J_{1-} + J_{2-}$ on the left- and right-hand side, respectively, to get the above state $|j = \frac{3}{2}, m = +\frac{1}{2}\rangle$. Iterating with the ladder operators gives the following states with $j = 3/2$. The state $|j = \frac{1}{2}, m = +\frac{1}{2}\rangle = a|m_1 = 0, m_2 = +\frac{1}{2}\rangle + b|m_1 = +1, m_2 = -\frac{1}{2}\rangle$, with $|a|^2 + |b|^2 = 1$, is then constructed to be orthogonal to $|j = \frac{3}{2}, m = +\frac{1}{2}\rangle$ already obtained. Then $J_- = J_{1-} + J_{2-}$ is applied again to obtain $|j = \frac{1}{2}, m = -\frac{1}{2}\rangle$.

Problem 4 (4 p.):

The energy levels for an ion with spin $s = 1$ in a crystal are determined by the Hamiltonian operator $H = H_0 + H_1$ where $H_0 = A S_z^2$ and $H_1 = B(S_x^2 - S_y^2)$ is a perturbation.

(a) Find the eigenvectors and corresponding energy eigenvalues to H_0 .

(b) Calculate the first order energy shifts due to H_1 and derive the corresponding zeroth order eigenvectors to H .

Solution:

(a) $|s = 1, m = 0, \pm 1\rangle$ are eigenstates to S^2 and S_z . Thus $H_0|1, m\rangle = A S_z^2|1, m\rangle = A(m\hbar)^2|1, m\rangle$, i.e. energy eigenvalue 0 for $|1, 0\rangle$ (ground state) and $A\hbar^2$ for the two degenerate states $|1, \pm 1\rangle$ (first excited states).

(b) Perturbation $H_1 = B(S_x^2 - S_y^2) = B(S_+^2 + S_-^2)/2$, since $S_{\pm} = S_x \pm iS_y$. The non-degenerate ground state has first order energy shift $\Delta_0^{(1)} = \frac{B}{2}\langle 1, 0|(S_+^2 + S_-^2)|1, 0\rangle = 0$, since S_{\pm}^2 attempts to shift the state up/down in m by two units to $m = \pm 2$ but there are no such states ($m = 0, \pm 1$ only). The state $|1, 0\rangle = 0$ is zeroth order approximation eigenvector to H .

For the degenerate states $|1, \pm 1\rangle$, perturbation theory gives the matrix equation

$$\frac{B}{2} \begin{pmatrix} \langle 1, +1|(S_+^2 + S_-^2)|1, +1\rangle & \langle 1, +1|(S_+^2 + S_-^2)|1, -1\rangle \\ \langle 1, -1|(S_+^2 + S_-^2)|1, +1\rangle & \langle 1, -1|(S_+^2 + S_-^2)|1, -1\rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \Delta_{\ell}^{(1)} \begin{pmatrix} a \\ b \end{pmatrix}$$

in terms of the first order energy shift $\Delta_{\ell}^{(1)}$ and the zeroth order eigenstates

$$|\ell^{(0)}\rangle = |1, +1\rangle\langle 1, +1|\ell^{(0)}\rangle + |1, -1\rangle\langle 1, -1|\ell^{(0)}\rangle \doteq \begin{pmatrix} \langle 1, +1|\ell^{(0)}\rangle \\ \langle 1, -1|\ell^{(0)}\rangle \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Repeated use of $J_{\pm}|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$ in the matrix equation gives

$$\begin{pmatrix} -\Delta_{\ell}^{(1)} & B\hbar^2 \\ B\hbar^2 & -\Delta_{\ell}^{(1)} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0. \text{ The secular equation } \det \begin{pmatrix} -\Delta_{\ell}^{(1)} & B\hbar^2 \\ B\hbar^2 & -\Delta_{\ell}^{(1)} \end{pmatrix} = 0 \text{ then gives}$$

$$\Delta_{\ell}^{(1)} = \pm B\hbar^2, \text{ which inserted in the matrix equation gives } B\hbar^2 \begin{pmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

with solutions $a = \pm b$, respectively for the positive/negative energy shift. The normalisation condition $|a|^2 + |b|^2 = 1$ then gives $a = \pm b = \frac{1}{\sqrt{2}}$, i.e. $|\ell^{(0)}\rangle = \frac{1}{\sqrt{2}}(|1, +1\rangle \pm |1, -1\rangle)$.

Summary: First order energy shift corresponding zeroth order eigenstate to H

$$\begin{array}{ll} \Delta_0^{(1)} = 0 & |1, 0\rangle \\ \Delta_{\pm 1}^{(1)} = \pm B\hbar^2 & \frac{1}{\sqrt{2}}(|1, +1\rangle \pm |1, -1\rangle) \end{array}$$

Problem 5 (4 p.):

Consider the elastic scattering of particles with mass m and energy $E = \hbar^2 k^2 / 2m$, on the repulsive spherically symmetric δ -shell potential $V(r) = A\delta(r - R)$, with $A > 0$ and $R > 0$.

(a) Calculate the differential cross section in Born approximation.

(b) Take the low-energy limit of the scattering amplitude and use this to calculate the scattering length in Born approximation.

Solution:

(a) The scattering amplitude in Born-approximation is given by

$$f_k^{(1)}(\theta, \varphi) = -\frac{4\pi^2 m}{\hbar^2} \langle \vec{k}' | V | \vec{k} \rangle = -\frac{2m}{\hbar^2} \frac{1}{4\pi} \int d^3 x' e^{i(\vec{k} - \vec{k}') \cdot \vec{x}'} V(\vec{x}')$$

Rewriting in polar coordinates with $\vec{q} = \vec{k} - \vec{k}'$ along z' -axis and $V(r) = A\delta(r - R)$ gives

$$\begin{aligned} f_k^{(1)}(\theta, \varphi) &= -\frac{2m}{\hbar^2} \frac{1}{4\pi} \int_0^\infty dr' r'^2 \int_{-1}^1 d(\cos \theta') \int_0^{2\pi} d\varphi' e^{iqr' \cos \theta'} A\delta(r' - R) = \\ &= -\frac{2m}{\hbar^2} \frac{1}{4\pi} A 2\pi R^2 \int_{-1}^1 d(\cos \theta') e^{iqR \cos \theta'} = -\frac{2mA}{\hbar^2} R^2 \frac{\sin qR}{qR} \end{aligned}$$

where $q = 2k \sin \theta / 2$ is the momentum transfer for elastic scattering.

The differential cross-section becomes

$$\frac{d\sigma^{(1)}}{d\Omega} = |f_k^{(1)}(\theta, \varphi)|^2 = \left(\frac{2mA}{\hbar^2} \right)^2 R^4 \left(\frac{\sin qR}{qR} \right)^2$$

(b) In the low-energy limit $k \rightarrow 0$ (remember $E = \frac{\hbar^2 k^2}{2m}$) and therefore $q \rightarrow 0$, one obtains

$$\lim_{k \rightarrow 0} f_k^{(1)}(\theta, \varphi) = \lim_{q \rightarrow 0} -\frac{2mA}{\hbar^2} R^2 \frac{\sin qR}{qR} = -\frac{2mA}{\hbar^2} R^2$$

The scattering length is given by $a \equiv a_0 = -\lim_{k \rightarrow 0} \frac{\tan \delta_0}{k}$ where the phase-shift δ_0 is related to the partial wave amplitude as $f_0(k) = \frac{1}{k \cot \delta_0 - ik}$ or equivalently $\frac{\tan \delta_0}{k} = \frac{1}{1/f_0 + ik}$. Thus we need to calculate the partial wave amplitude for S-wave scattering $f_0^{(1)}$ in the Born approximation. The partial wave expansion of a scattering amplitude is given by $f_k(\theta, \varphi) = \sum_l (2l+1) f_l P_l(\cos \theta)$ and thus $f_0^{(1)}$ is obtained by multiplying both sides with $P_0(\cos \theta) = 1$ and integrating over $\cos \theta$ exploiting the orthogonality of the Legendre polynomials, $\int_{-1}^1 d(\cos \theta) f_k^{(1)}(\theta, \varphi) = \int_{-1}^1 d(\cos \theta) \sum_l (2l+1) f_l P_l(\cos \theta) = 2f_0^{(1)}$, which gives (still in the low-energy limit)

$$\lim_{k \rightarrow 0} f_0^{(1)} = \frac{1}{2} \int_{-1}^1 d(\cos \theta) \lim_{k \rightarrow 0} f_k^{(1)}(\theta, \varphi) = -\frac{2mA}{\hbar^2} R^2$$

Finally we get the scattering length in Born approximation,

$$a^{(1)} = -\lim_{k \rightarrow 0} \frac{\tan \delta_0^{(1)}}{k} = -\lim_{k \rightarrow 0} f_0^{(1)} = \frac{2mA}{\hbar^2} R^2$$