

UPPSALA UNIVERSITY  
Dept. of Radiation Sciences  
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Examination 2006-10-27 in  
**Quantum Mechanics, advanced course**  
(Kvantmekanik fk, 5 poäng, 1TT173, F4T)  
Engineering Physics (teknisk fysik)

Time: 9–14, i.e. 5 hours, at Magistern

Allowed aids: Physics Handbook, Mathematics Handbook,  
Taschenbuch der Mathematik,  
enclosed collection of formulae, calculator.

Instructions: - write legible (skriv läsligt), define symbols, motivate equations etc.  
- write your name on each sheet, at most one problem per sheet  
- put your solutions in number order in this cover

*Good luck !*

**Note:**

Results available on [www3.tsl.uu.se/thep/courses/QM/](http://www3.tsl.uu.se/thep/courses/QM/) latest on Thursday November 2  
Exams shown and given back Thursday November 2, during 12.00-12.45 in Å2004

Name:

Personal number (personnummer):

Program (e.g. T, mat.nat.):

Year of registration:

Number of sheets per problem:

1	2	3	4	5
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Results:

1	2	3	4	5
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Sum:

Grade:

1. (a) Define the concept of unitary operator and show that the operator  $e^{ipa/\hbar}$ , where  $p$  is the momentum operator in one dimension, is unitary.  
 (b) Show that  $e^{ipa/\hbar}|x'\rangle$  is an eigenstate to the position operator. What is the eigenvalue? (Hint: Consider  $[x, e^{ipa/\hbar}]$ .)  
 (4 p.)
  
2. A one-dimensional harmonic oscillator is at time  $t = 0$  in the state  $|\alpha, t = 0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ , where  $|n\rangle$  are the energy-eigenstates with energy eigenvalues  $E_n = \hbar\omega(n + 1/2)$ .  
 (a) Give the state  $|\alpha, t\rangle$  at a later time  $t$ .  
 (b) For this state, determine the expectation value of the kinetic energy  $E_k = p^2/2m$  as a function of time.  
 (4 p.)
  
3. A physical system contains two particles with angular momenta  $\vec{J}_1$  and  $\vec{J}_2$ .  
 (a) Give the relation between the two bases which is given by the eigenvectors to the internally commuting operators  $\vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}$  (direct product basis) and the operators  $\vec{J}^2, \vec{J}_z$  where  $\vec{J} = \vec{J}_1 + \vec{J}_2$ .  
 (b) Show that the Clebsch-Gordan coefficients are non-zero only if  $m = m_1 + m_2$  where  $m\hbar, m_1\hbar, m_2\hbar$  are the eigenvalues to  $J_z, J_{1z}, J_{2z}$ .  
 (c) Let the two particles have angular momenta 1 and 1/2, respectively, which are added to a total angular momentum  $j$ . Give (with motivation) all resulting states  $|j, m\rangle$  expressed in  $|j_1 = 1, m_1\rangle$  and  $|j_2 = 1/2, m_2\rangle$ .  
 (4 p.)
  
4. The energy levels for an ion with spin  $s = 1$  in a crystal are determined by the Hamiltonian operator  $H = H_0 + H_1$  where  $H_0 = A S_z^2$  and  $H_1 = B(S_x^2 - S_y^2)$  is a perturbation.  
 (a) Find the eigenvectors and corresponding energy eigenvalues to  $H_0$ .  
 (b) Calculate the first order energy shifts due to  $H_1$  and derive the corresponding zeroth order eigenvectors to  $H$ .  
 (4 p.)
  
5. Consider the elastic scattering of particles with mass  $m$  and energy  $E = \hbar^2 k^2/2m$ , on the repulsive spherically symmetric  $\delta$ -shell potential  $V(r) = A\delta(r - R)$ , with  $A > 0$  and  $R > 0$ .  
 (a) Calculate the differential cross section in Born approximation.  
 (b) Take the low-energy limit of the scattering amplitude and use this to calculate the scattering length in Born approximation.  
 (4 p.)

**Collection of formulae**

Angular momentum:  $J_{\pm} = J_x \pm iJ_y$        $J_{\pm}|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$

Addition:  $|j_1, j_2; j = j_1 + j_2, m = j\rangle = |j_1, j_2; m_1 = j_1, m_2 = j_2\rangle$

$[x_i, F(\vec{p})] = i\hbar\frac{\partial F}{\partial p_i}$  ;  $[p_i, G(\vec{x})] = -i\hbar\frac{\partial G}{\partial x_i}$

Pauli matrices:  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$        $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$        $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Harmonic oscillator:  $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

$a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + \frac{ip}{m\omega} \right)$        $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{ip}{m\omega} \right)$        $[a, a^\dagger] = 1$        $N = a^\dagger a$

$a|n\rangle = \sqrt{n}|n-1\rangle$        $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

Time-independent perturbation theory:

Non-degenerate eigenvalue

$|n\rangle = |n^{(0)}\rangle + \lambda \sum_{k \neq n} |k^{(0)}\rangle \frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} + \dots$  ;  $\Delta_n \equiv E_n - E_n^{(0)} = \lambda V_{nn} + \lambda^2 \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \dots$

Degenerate eigenvalue

$|\ell\rangle = |\ell^{(0)}\rangle + \lambda|\ell^{(1)}\rangle + \dots$        $\Delta_\ell = \lambda\Delta_\ell^{(1)} + \lambda^2\Delta_\ell^{(2)} + \dots$

$$\begin{pmatrix} V_{11} & V_{12} & \cdots \\ V_{21} & V_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \langle 1^{(0)} | \ell^{(0)} \rangle \\ \langle 2^{(0)} | \ell^{(0)} \rangle \\ \vdots \end{pmatrix} = \Delta_\ell^{(1)} \begin{pmatrix} \langle 1^{(0)} | \ell^{(0)} \rangle \\ \langle 2^{(0)} | \ell^{(0)} \rangle \\ \vdots \end{pmatrix}$$

Time dependent perturbation theory:

$$\begin{aligned} c_n^{(0)}(t) &= \delta_{ni} \\ c_n^{(1)}(t) &= \frac{-i}{\hbar} \int_{t_0}^t \langle n | V_I(t') | i \rangle dt' = \frac{-i}{\hbar} \int_{t_0}^t e^{i\omega_{ni}t'} V_{ni}(t') dt' \\ \omega_{ni} &= \frac{E_n - E_i}{\hbar} \end{aligned}$$

The Golden Rule:  $w_{i \rightarrow n} = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i)$

The scattering amplitude:  $f(\mathbf{k}', \mathbf{k}) = -\frac{2m}{\hbar^2} \frac{(2\pi)^3}{4\pi} \langle \mathbf{k}' | T | \mathbf{k} \rangle \stackrel{V=V(r)}{=} \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos \theta)$

Perturbative expansion:  $T = V + VG^+V + \dots$

Partial wave amplitude:  $f_l(k) = \frac{1}{2ik} (e^{2i\delta_l} - 1) = \frac{1}{k} e^{i\delta_l} \sin \delta_l = \frac{1}{k \cot \delta_l - ik}$

Total elastic scattering cross section:  $\sigma_{tot} = \int |f(\mathbf{k}', \mathbf{k})|^2 d\Omega \stackrel{V=V(r)}{=} 4\pi \sum_{l=0}^{\infty} (2l+1) |f_l(k)|^2$

Plane wave:  $\langle \mathbf{x} | \mathbf{k} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}$  ,  $e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}})$

Clebsch-Gordan coefficients and spherical harmonics

Note: A square-root sign is to be understood over every coefficient, e.g., for  $-8/15$  read  $-\sqrt{8/15}$ . Notation:  $\begin{matrix} J & J & \dots \\ M & M & \dots \end{matrix}$

$1/2 \times 1/2$   $\begin{matrix} 1 & & & & \\ +1 & 1 & 0 & & \\ +1/2 +1/2 & 1 & 0 & 0 & \\ +1/2 -1/2 & 1/2 & 1/2 & 1 & \\ -1/2 +1/2 & 1/2 & -1/2 & -1 & \\ -1/2 -1/2 & & & & 1 \end{matrix}$

$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$

$2 \times 1/2$   $\begin{matrix} 5/2 & & & & \\ +5/2 & 5/2 & 3/2 & & \\ +2 & 1/2 & 1 & 3/2 +3/2 & \\ +2 -1/2 & 1/5 & 4/5 & 5/2 & 3/2 \\ +1 +1/2 & 4/5 & -1/5 & +1/2 & +1/2 \end{matrix}$

$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$

$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$

$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$

$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$

$3/2 \times 1/2$   $\begin{matrix} 2 & & & & \\ +2 & 2 & 1 & & \\ +3/2 +1/2 & 1 & +1 & +1 & \\ +3/2 -1/2 & 1/4 & 3/4 & 2 & 1 \\ +1/2 +1/2 & 3/4 & -1/4 & 0 & 0 \end{matrix}$

$3 \times 1/2$   $\begin{matrix} 3 & & & & \\ +3 & 3 & 2 & & \\ +2 +1 & 1 & +2 & +2 & \\ +2 & 0 & 1/3 & 2/3 & 3 & 2 & 1 \\ +1 & +1 & 2/3 & -1/3 & +1 & +1 & +1 \end{matrix}$

$3/2 \times 1$   $\begin{matrix} 5/2 & & & & \\ +5/2 & 5/2 & 3/2 & & \\ +3/2 +1 & 1 & +3/2 +3/2 & & \\ +3/2 & 0 & 2/5 & 3/5 & 5/2 & 3/2 & 1/2 \\ +1/2 +1 & 3/5 & -2/5 & +1/2 & +1/2 & +1/2 & \end{matrix}$

$1 \times 1$   $\begin{matrix} 2 & & & & \\ +2 & 2 & 1 & & \\ +1 +1 & 1 & +1 & +1 & \\ +1 & 0 & 1/2 & 1/2 & 2 & 1 & 0 \\ 0 & +1 & 1/2 & -1/2 & 0 & 0 & 0 \end{matrix}$

$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$

$d_{m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$

$\begin{matrix} \langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle \\ = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M \rangle \end{matrix}$

Spherical Bessel and Neumann functions

$$j_l(kr) \xrightarrow{r \rightarrow \infty} \frac{1}{2ik} \left[ \frac{e^{i(kr-l\pi/2)}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r} \right]$$

$$\begin{matrix} j_l(kr) & \xrightarrow{kr \rightarrow 0} & \frac{2^l l!}{(2l+1)!} (kr)^l & j_0(kr) & = & \frac{\sin(kr)}{kr} \\ n_l(kr) & \xrightarrow{kr \rightarrow 0} & -\frac{(2l)!}{2^l l!} \frac{1}{(kr)^{l+1}} & n_0(kr) & = & -\frac{\cos(kr)}{kr} \end{matrix}$$

Scattering length:  $a \equiv a_0 = -\lim_{k \rightarrow 0} \frac{\tan \delta_0}{k}$

For inelastic scattering we have  $d\sigma(0 \rightarrow n) = w_{\mathbf{k},0 \rightarrow [\mathbf{k}' \in d\Omega, n]} / j_{in}$  where the transition rate is given by

$$w_{\mathbf{k},0 \rightarrow [\mathbf{k}' \in d\Omega, n]} = \frac{2\pi}{\hbar} |\langle \mathbf{k}', n | V | \mathbf{k}, 0 \rangle|^2 \left( \frac{L}{2\pi} \right)^3 \left( \frac{mk'}{\hbar^2} \right) d\Omega$$

at box normalisation,  $\langle \mathbf{x} | \mathbf{k}, n \rangle = \frac{1}{L^{3/2}} e^{i\mathbf{k} \cdot \mathbf{x}} |n\rangle$ .

$$\int d^3x \frac{e^{i\mathbf{q} \cdot \mathbf{x}}}{r} = \frac{4\pi}{q^2}$$